STABILITY IN MATCHING MARKETS WITH PEER EFFECTS

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Abstract. The paper investigates conditions which guarantee the existence of a stable outcome in a school matching in the presence of peer effects. We consider an economy where students are characterized by their type (e.g. SAT score), and schools are characterized by their value (e.g. teaching quality) and capacity. Moreover, we divide students and schools into groups, so that going to a school outside of one's group may be associated with additional costs or even prohibited. A student receives utility from a school per se (its value minus costs of attending) and from one's peers, students who also go to that school. We find that sufficient condition for a stable matching to exist is that a directed graph, which governs the possibility to go from one group to another, should not have (directed or undirected) cycles. We also construct a polynomial time algorithm, which produces a stable matching. Furthermore, we show that if the graph has a cycle, then there exist other economy parameters (types, costs and so on), so that no stable matching exists. In addition, in cases where a stable matching exists we investigate whether it is unique.

1. Introduction

Peer effects are a common phenomenon in everyday life. Parents often try to place their kids in schools where they believe that their children would have good classmates. That is, parents care not only about quality of teachers and curriculum, but also about whom is going to study with their children. Similarly, many students want to go to Ivy League universities because of the connections that they are likely to make at such places.

The presence of peer effects in schooling was noticed more than fifty years ago (see e.g. Coleman et al. (1966, Section 2.4)). Sacerdote (2011) provides an overview of the current state of empirical research on peer effects and points to its importance. A number of recent

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Date: October 6, 2019.

The author would like to thank Larry Samuelson for valuable comments, suggestions, and encouragement throughout the duration of the project. The author is grateful to Vadim Gorin for support and fruitful discussions. The paper benefited from comments and suggestions by anonymous referees and seminar audiences at Yale and UW Madison.

papers show the significance of peer effects in schooling (yet, the magnitude of the importance of peer effects varies across papers). Examples are Ding and Lehrer (2007) (peer effects in China), Sacerdote (2001) (peer effects at Dartmouth), Winston and Zimmerman (2004) (peer effects in US colleges), Zabel (2008) (peer effects in New York), Zimmerman (2003) (peer effects at Williams College). For more examples and details see also Epple and Romano (2011), which provides a survey of recent papers on various types of peer effects in education.

When we go to theory, the relationship between schools or colleges and students is usually modelled as a two-sided matching problem. In matching models without peer effects and externalities, substitutability is a sufficient (and in some sense necessary) condition for the existence of a (group) stable matching (see Gale and Shapley (1962), Hatfield and Milgrom (2005) and Hatfield and Kojima (2008)). Unfortunately, matching models with peer effects are known to often lack existence of equilibria. This motivates us to study theoretical models of matching in the presence of peer effects.

Several authors investigated the existence of stable matchings in the presence of peer effects in the recent years. Two prevalent approaches are either doing an algorithmic search for an equilibrium (e.g. Echenique and Yenmez (2007)) or imposing some structural restrictions on a model, which lead to an automatic existence of an equilibrium (e.g. Bodine-Baron et al. (2011), Pycia (2012)). These and other related articles will be discussed in more detail below. Despite the progress, there is still no simple criterion for the existence of stable matchings, which would apply to wide enough class of models. From the theoretical viewpoint, one would like to find out what basic features of preferences can guarantee existence (or non-existence) of an equilibrium in the presence of peer effects.

Our paper makes a step in this direction. We do not look at the most general choices of preferences, as it is hardly possible to say anything at all if no structure is present in the model (see, however, Section 7 for some generalizations). Instead, we propose and analyze a model which captures two main features: utility of a student depends on peers matched to the same school through a (quite general) peer effect function, and schools have intrinsic qualities and are arranged in districts, with moving costs arising when a student wants to change his district. Many authors agree that both peer effects and subdivision into districts have to be present in any realistic school matching model (see, e.g., Calsamiglia et al. (2015)

and references therein), and, therefore, it naturally leads to studying the effect of these two basic features on the existence of an equilibrium.

We start by modifying the college admission model, which was studied in the seminal paper of Gale and Shapley (1962), and add preferences over schoolmates. Consequently, students now care both about their assigned school and their peers. This is modelled as a linear combination of school-related utility and utility from a given set of peers. For most of the paper we focus on pairwise stable matchings; in the context of schools this means that no single student can profitably deviate to another school which would accept him. We believe pairwise stability to be a natural assumption in case of schools, where a parent cannot easily coordinate with other parents and place their children in the same school. However, we also discuss group stability in Section 8.

The division of schools into districts is based on the following real life phenomena. Sometimes a person may be prohibited from applying to particular schools. For example, religious schools generally accept only those students who practice the same religion. Furthermore, to go to a Jewish school, one often needs to present a proof of one's Jewish roots. Similarly sometimes schools accept only those who live in specified areas. Thus, students, who live outside of those areas cannot be admitted. Public schools in the US, which typically accept only students living in their school zones, are a natural example of the above. Moreover, a large set of schools in Moscow function in that way, as well as schools in many other countries/cities. Finally, segregation corresponds to the structure, where some students are restricted from some set of schools. Instead of schools we can also think about specific majors. Then it may be too late (and, thus, impossible) to switch from studying, e.g., ballet to studying quantum physics. Those patterns can be encoded into a graph. Possibility/impossibility to move from one group to the other corresponds to the presence/absence of an edge between the groups, which correspond to graph vertices.

Let us describe the **general setting** in more detail. In our model preferences of schools coincide: they prefer students who have higher type (e.g. higher test scores). We allow more flexibility on the students' side. We divide the set of students into groups and, similarly, we assign each school to one of those groups. All students from the same group have the same valuations of schools.

Such division can be viewed as different markets. That is, being in one group means being from the same market such as country, race, religion, specialization, etc. A school attached to a group is located in the same market as students from that group. For example, they all are in the same city. Then the difference between how a student from a market values a school from the same market and how a student from a different market values that school is expressed as an additional "market switching" cost c. Such cost is location and origin specific. We can view this cost as the expenses associated with buying an apartment near a school or with commuting costs or with costs of switching from one field of primary study to the other (e.g. switching from mathematically inclined school to the one which focuses more on humanities).

To sum up, we get a set of separate markets, where students only differ by their ability or type. Going to a school in a foreign market is associated with additional costs for a student born in a different home market. Obviously, in some cases such cost may be prohibitively high, so that there is no way a student from a market X can attend a school in a market Y (e.g. religious schools for someone outside of that religion or legal segregation of schools in the US in the 20th century). We can summarize that prohibition by drawing an oriented graph, where vertices represents our groups/markets, and an edge from one vertex to another means that switch from the former to the latter is not prohibited. Prohibitively high costs will be crucial for our results. What would matter for our constructions and conclusions is the oriented switching graph, and not the exact values of intermediate or not prohibitive switching costs.

Our main result provides conditions for the existence of a stable matching in the above model. We find that the sufficient condition is that there are no cycles (neither directed, nor undirected) in the directed graph of possible market switches: when the switching graph is an oriented forest, we present an algorithm, which produces a stable matching. Further, we show the necessity of "no cycles" condition: if there is a cycle (possibly undirected), then there are parameters for which there is no stable matching. We refer to Section 2 for an illustrative example of the importance of acyclicity condition in the simplest case when the graph has two vertices. Our main results are given in Theorems 1 and 2. We also discuss when a stable matching is unique/non-unique (see Propositions 4 and 5).

To our knowledge, the most novel aspect of our condition lies in the non-directness. The classical results on the existence of a stable matching prohibit only directed cycles. E.g. in a roommate problem (Gale and Shapley (1962)) lack of directed cycles in agents' preferences guarantees stability. Non-directed cycles were not playing a major role before, however, for our setting they are of the same importance as directed cycles.

Related papers, which investigate the existence of stable matching with peer effects, are Pycia (2012), Echenique and Yenmez (2007), and Bodine-Baron et al. (2011). The first paper provides a condition (pairwise alignment of preferences), which guarantees the existence of a core stable (and, thus, also pairwise stable) matching. This condition and ours are non-nested. In our setting, pairwise alignment means that if we assign two students, say a and b, to some school with some set of peers and then consider a different assignment, where again a and b are at the same school, they must agree on whether the former or the latter allocation is better. However, such condition is not satisfied in our framework: a and b may disagree even if they were born in the same market, because they have different set of peers (a is in the set of peers of b, but not in the set of peers of itself), and this distinction may be of different importance depending on how large a school is. When a school is small having one better peer means more than when a school is large. So that even if the quality per se of a smaller school is worse, a may still prefer it: e.g. if b is a very good peer, a may want to choose a small school, where there will be almost no one except itself and b. But if the second school is much better than the first, and is filled with students similar to a, b may choose the second, larger school. Thus, preferences are not aligned, and our model is not a subset of what Pycia (2012) considers. For a more detailed discussion, see Section 4.3. The second paper, Echenique and Yenmez (2007), presents an algorithm, which produces a set of allocations containing all stable matchings in case they exist. However, it requires searching for a fixed point of a certain operator over the set of all possible matchings, and implementing such an algorithm may be very time consuming (in fact, in some cases it leads to just checking all possible allocations). In contrast, we provide specific conditions for a stable matching to exist, so that we do not need to check different possible allocations. It would be interesting to study whether structural restrictions similar to the ones considered in our text can lead to the significant decrease of the running time of the algorithm of Echenique and Yenmez (2007). The third paper, Bodine-Baron et al. (2011), suggests a model for matching in the presence of peer effects arising from an underlying social network. In their model a properly defined equilibrium always exists. However, the key assumption guaranteeing this is a certain symmetry condition for the social network graph — no analogue of this condition is present in our setting.

Other related literature

One of the important aspects of our setting is subdivision into districts. The idea that sometimes agents have to choose from subsets of possible matching partners (i.e. they belong to a smaller market with fixed subset of alternatives) or there is a separate matching technology inside each market has been present in the market design literature. Though, the most common question differs from ours: the literature mostly concerns merging two markets into one and analyzing whether it can be beneficial for agents.

For example, Ortega (2018) considers the matching between men and women when agents are partitioned into disjoint groups. The question is then whether there is a way to improve the inside-groups matching by allowing agents to match outside of their groups. He shows that a stable matching across the whole population cannot hurt more than half of the society compared to the inside-groups outcome. Doğan and Yenmez (2017) investigate the market, where only one side, schools, is divided into groups. Students are not restricted to a specific group and can apply to any school. Each group of schools runs a separate matching algorithm, so that at the end some students get offers from multiple groups, while some do not get any offers at all. The authors propose a unified enrollment system which turns out to be better for students. It works jointly with all schools and leads to at most one offer to each student. Manjunath and Turhan (2016) propose an alternative approach to improve upon the outcome of the independent across school groups admission process. They suggest to do an iterative rematching after the independent processes are done.

Similarly, Nikzad et al. (2016) look at the possibility of merging two markets into one in the context of kidney exchange. One of the markets represents the US, where kidney exchange is well developed, but there is lack of donors, and the other represents a developing country with almost no suitable medical facilities, but with willing donors. The authors show that merging those markets into one will increase the welfare in the US.

On the peer effect side, the coalition formation literature such as Bogomolnaia and Jackson (2002), Banerjee et al. (2001), and Kaneko and Wooders (1986) is relevant. If one views schools as additional agents and asks players to form coalitions, additionally assuming that coalition with more than one or zero schools will lead to a utility of a negative infinity, we get precisely the problem of finding a stable coalition. However, our model does not satisfy conditions from the above papers to ensure existence of a stable matching.

The rest of the paper is organised as follows. Section 2 presents a motivating example, which illustrates the main ideas of the paper. Section 3 builds up the model and defines our solution concept, pairwise stability. Section 4 provides the sufficient condition (no cycles) for the existence of a pairwise stable matching, while Section 5 shows that that condition can be viewed as necessary: for any graph G with a cycle there exist set of other parameters (types, school values, etc.), so that in the corresponding economy no stable matching exists. Section 6 talks about uniquenes/non-uniqueness of a stable matching, when it exists. Section 7 discusses the role, which our assumptions play in obtaining the results, and possible generalizations. Section 8 discusses alternative stibility definitions. Finally, Section 9 concludes. All proofs are in the Appendix.

2. Illustrative example

In this section we consider an example, which illustrates the model and the associated existence problem.

Suppose we have two schools, $\mathcal{A}$ and $\mathcal{B}$, and each school has two seats. There are four students characterized by their type (e.g. test score) $\theta = 0, 7, 8, 10$. Schools prefer students with higher types, and utility of a student θ sharing a school s with another student θ' is

$$u_\theta(s, \theta') = v_\theta(s) + \theta'.$$

If θ is alone at school s, then $u_\theta(s, \emptyset) = v_\theta(s)$. Utility of the school per se, $v_\theta(s)$, is

$v_\theta(s)$:

$\theta \backslash s$	$\mathcal{A}$	$\mathcal{B}$
0, 10	10	5.5
7, 8	6	9.5

In particular, 0 and 10 prefer school $\mathcal{A}$, while 7 and 8 prefer school $\mathcal{B}$. The interpretation is that 0 and 10 live in the district of school $\mathcal{A}$, and therefore, going to $\mathcal{B}$ leads to additional switching costs, thus, decreasing utility. Similarly, 7 and 8 live in the district of school $\mathcal{B}$.

The example is in some sense similar to a classical roommate problem with two rooms and four students, one of whom no one likes (Gale and Shapley (1962)). Here we have a zero type, whom no one wants as a peer, as it means zero peer effects. Although 10 is the best possible peer, it is still not worth to switch to a less desirable school to join 10, if the most favorite one has a "normal" (i.e. 7 or 8) peer.

There are no pairwise stable matchings, which means that in any matching there is a student who can profitably deviate to a different school and this school will admit him/her. The argument, summarized in Figure 1, is:

- If $(8,10) \to \mathcal{A}$, then $7 \to \mathcal{B}$, so that 8 deviates to $\mathcal{B}$:

$$u_8(\mathcal{A},10) = 10 + 6 = 16 < 9.5 + 7 = 16.5 = u_8(\mathcal{B},7);$$

- Similarly, if $(7,10) \to \mathcal{A}$, then 7 deviates;
- If $(8,10) \to \mathcal{B}$, then $7 \to \mathcal{A}$, so that 10 deviates to $\mathcal{A}$:

$$u_{10}(\mathcal{B},8) = 5.5 + 8 = 13.5 < 10 + 7 = 17 = u_{10}(\mathcal{A},7);$$

- Similarly, if $(7,10) \to \mathcal{B}$, then 10 deviates;
- If $(7,8) \to \mathcal{B}$, then $10 \to \mathcal{A}$, so that 10 deviates to $\mathcal{B}$:

$$u_{10}(\mathcal{A},0) = 10 + 0 = 10 < 5.5 + 8 = 13.5 = u_{10}(\mathcal{B},8);$$

- Similarly, if $(7,8) \to \mathcal{A}$, then 10 deviates;

Thus, there are no stable matchings in the above economy. Our example corresponds to a graph with two verices representing the markets of schools $\mathcal{A}$ and $\mathcal{B}$ and two directed edges, $\{\mathcal{A} \to \mathcal{B}\}$ and $\{\mathcal{B} \to \mathcal{A}\}$. This graph has a cycle.

Now suppose that it is prohibitively costly for student 10 to attend school $\mathcal{B}$, perhaps because student 10 is the wrong religion or gender, or faces prohibitive relocation costs. Then the existence of a stable matching is restored. We assign $0, 10$ to $\mathcal{A}$ and $7, 8$ to $\mathcal{B}$. 10 is not allowed to deviate, and we get a stable matching, as summarized in Figure 2. This stable matching can be obtained as an outcome of the algorithm presented in Section 4.1. That section also shows step-by-step how the algorithm leads to the allocation $\{0,10\} \to \mathcal{A}$, $\{7,8\} \to \mathcal{B}$.

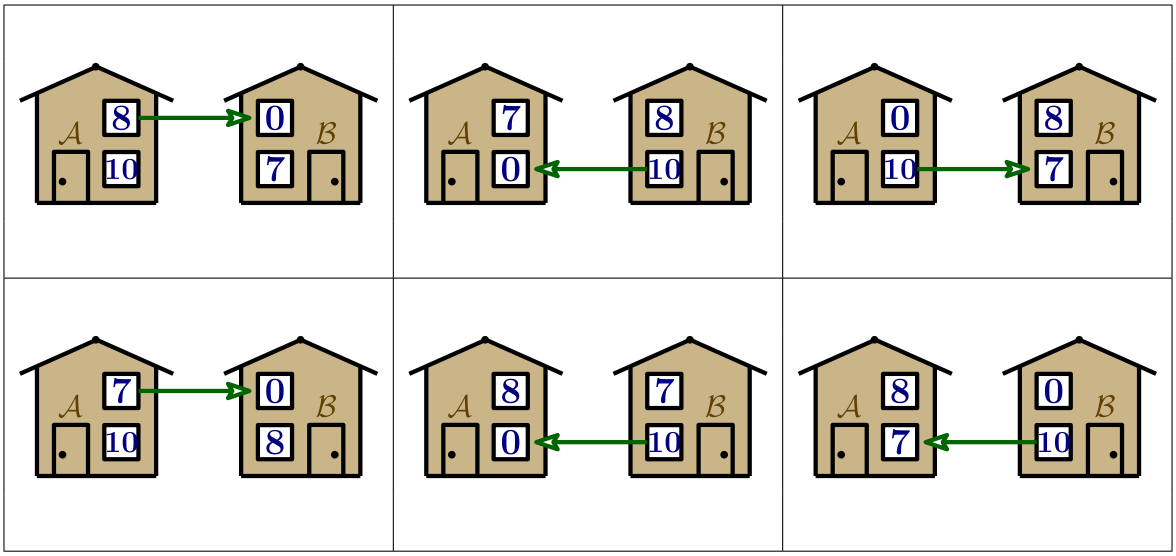

FIGURE 1. No stable matchings in the illustrative example. Arrows represent profitable deviations.

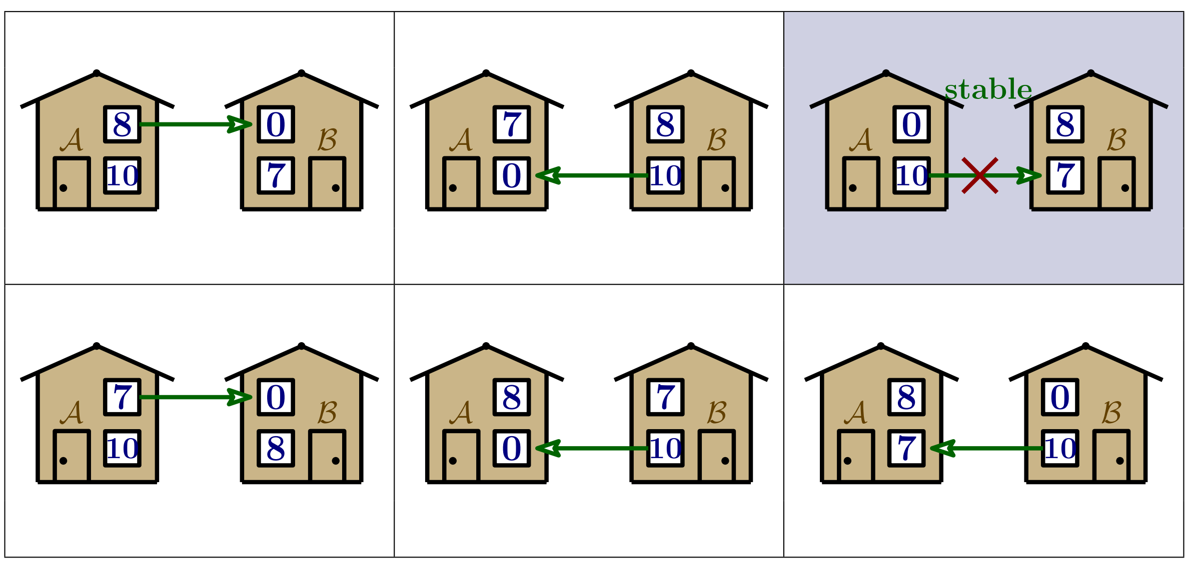

FIGURE 2. Stable matching in the illustrative example.

On the graph side, the prohibitive cost means that our graph has a directed edge from the market of school $\mathcal{B}$ to the market of school $\mathcal{A}$, but not in the opposite direction. This graph has no cycle.

3. Basic model

3.1. **Setting.** Let us consider a world with n markets. Each market i has k_i different schools. In any market i, any school ℓ has capacity[1] $q_\ell^i \geq 2$ and is associated with the utility v_ℓ^i. Each school cannot exceed its capacity for students and would like to take as many students below capacity as possible. Moreover, schools prefer students with higher ability.[2] Without loss of generality we number schools in each market by their attractiveness, i.e. we assume $v_1^i \geq v_2^i \geq \ldots \geq v_{k_i}^i$ for all $i = 1, \ldots, n$. Without loss of generality we may also assume that the best school is located in market 1, that is we assume $v_1^1 \geq v_1^i$ for all $i = 2, \ldots, n$. To shorten the notation, we also sometimes use v_ℓ^i to denote the school itself: this is the school ℓ in the market i. When it is important to stress that we are referring to school's name, not its attractiveness level, we will use the notation (ℓ, i)

Additionally, each market i is populated with d_i students of various abilities[3]. Changing one's initial market is costly for the students. The possibility to switch between markets is governed by a directed **switching graph** $G = (V, E)$, where markets are vertices. That is, the graph G consists of n vertices ($|V| = n$). Edges represent the possibility to switch from one market to another. If $\{i \to j\} \in G$, then it is allowed to switch from market i to market j, although the switch may be associated with some costs. If $\{i \to j\} \notin G$, then it means that market j is infeasible to students born in market i. That is, either it is too costly for them to attend (even the best allocation in j would not offset switching costs) or it is prohibited by underlying laws. The interpretation of the switching graph G is discussed in the Introduction and at the end of Section 7.

Each student is characterized by type θ and home market i. Let F_i denote all types of students (with repetitions) in market i. Formally, each F_i is a multiset (see Definition 1). We assume that all types are non-negative and each F_i is finite and has a large enough number of zero types. Zero types represent the lowest ability students. Having

[1]We may allow for capacity of one with minor modification to the Algorithm in Theorem 1.

[2]This corresponds to maximizing the sum of the abilities of all admitted students.

[3]Most of our results also hold for continuous distributions, so that instead of d_i students we will have mass d_i of students. For example, the algorithm in Theorem 1 is still valid: if higher types do not deviate, then lower types also stay. Discreteness is only used in the construction in Theorem 2.

$z_i = \max_{j:\, j=i \text{ or } \{j \to i\} \in G} \max_{\ell=1,\dots,k_j} \left(q_\ell^j - 1\right)$ zeros in each market i is enough for the following Theorem 1 to hold, though, in practice we need even smaller number of zero types. Having such number of zeros guarantees that if a non-zero type from a market i goes to a school v_ℓ^j, then there are enough of zero types from the market i to join the student, so that there will be no empty seats at v_ℓ^j. This helps us to avoid partially filled schools with non-zero students.[4] They fill seats which would otherwise be empty. We discuss the zero types assumption and its role in more detail in Sections 7 and 8.

The difference across students in different markets comes from the fact that if a student from market i wants to change one's initial market and apply to a school in a different (but feasible, i.e. such that $\{i \to j\} \in G$) market j, the student has to bear additional cost $c_{ij} \geq 0$, where $c_{ii} = 0$. This can be viewed as travelling costs of going to a foreign market (e.g. additional time it takes every morning to go to a further located school). Alternatively we can view those costs as psychological losses from being far from one's family and/or being surrounded by people from a different background. This is in a sense a mismatch penalty. There can be a number of other interpretations of costs.

Each student has an outside option with 0 utility (i.e. not attend a school). If a student does attend a school, then one's utility from attending a school is composed of school's own effect v_ℓ^j and a peer effect. The peer effect is described by a peer-effect function $p(\cdot)$, defined on all non-negative real multisets. Our basic model assumes that the peer-effect function does not depend on the student's own type, however, we discuss some possible generalizations in this direction in Section 7.

Definition 1. *A* ***non-negative real multiset*** *is a finite collection of non-negative reals, in which we allow the same number to be repeated arbitrary many times. The order of the elements of the multiset is irrelevant.*

Let $m[\mathbb{R}_+]$ denote the family of all non-negative real multisets.

Definition 2. *A* ***peer-effect function*** *is a non-negative function* $p : m[\mathbb{R}_+] \to \mathbb{R}_+$*, such that* $p(\emptyset) = 0$ *and* $p(\cdot)$ *is increasing. By increasing we mean that for any multiset* Θ *if*

[4] In the algorithm proposed in Theorem 1 we assign seats to zero types when some school cannot be filled by strictly positive types. Zero types from that school's home market occupy the remaining seats at that school.

$\theta' \geq \sup\Theta$ ($\theta' \leq \inf\Theta$), then $p(\Theta \cup \{\theta'\}) \geq p(\Theta)$ ($p(\Theta \cup \{\theta'\}) \leq p(\Theta)$), and if $\theta' > \theta$, then $p(\Theta \cup \{\theta'\}) \geq p(\Theta \cup \{\theta\})$.

The intuition is that peers provide an extra source of information and they may help in learning. Thus, when there are no peers, there is no such extra source, and $p(\emptyset) = 0$. The better is the set of peers, the more they (peers) know and the more a student can learn from them. Therefore, we assume that peer-effect function is increasing.

For a student ϑ of type θ and home market i, by its peers we always mean the multiset of types of all other students attending the same school as ϑ.

Finally, utility for a student ϑ of type θ and home market i from attending a school v_ℓ^j is

$$u_\vartheta(v_\ell^j, \text{ peers of } \vartheta) = v_\ell^j + \alpha \cdot p(\text{peers of } \vartheta) - c_{ij},$$

where the coefficient $\alpha \geq 0$ measures the importance of peer effects. When it leads to no confusion, we use $u_{\theta,i}$ instead of u_ϑ.

The average quality of one's peers is the natural example of a peer-effect function. It satisfies our assumptions, and we will use it quite frequently. Moreover, in some sense it corresponds to an approach in empirical research where a student's outcome Y (e.g. a test score or alcohol use) linearly depends on the average of background characteristic (types in our case) of one's peers (see, for example, review article Sacerdote (2011)).

Denote by $s(\vartheta)$ the school, where student ϑ goes, and by Θ the multiset of types of all peers of that student. That is, $\Theta = \{\text{type of } \vartheta' \mid \vartheta' \neq \vartheta, \quad s(\vartheta) = s(\vartheta')\}$. Then the average quality of ϑ's peers is:

$$p(\Theta) = \frac{\sum\limits_{\theta' \in \Theta} \theta'}{\sum\limits_{\theta' \in \Theta} 1},$$

where term θ' appears in the sum as many times as it appears in the multiset Θ.

Two other common examples of a peer effect function are the best and the worst types: $p(\Theta) = \sup\Theta$ and $p(\Theta) = \inf\Theta$. Similarly, we can do an average of, say, two best or two worst students.

Our setting allows for complementarities on the students' side. Suppose a student has to choose among a set of schools and peers. Suppose also that the choice set includes a low quality school which is not chosen by the student. Yet when we add to the choice set a peer of a very high type, who can only go to that school (because of the switching graph restrictions), the best choice can become the low quality school with the high type peer. So the school and the high type peer are complements, and complements appear as a consequence of the existence of the switching graph.

3.2. **Stable matching.** We are interested in pairwise stable matchings, so that no student-school pair can profitably deviate and match together.

Definition 3. *A **matching** is a mapping μ from the set of all students into the set of all schools, such that for each school ℓ in each market i,*

$$|\mu^{-1}(\ell, i)| \leq q_\ell^i,$$

where $|M|$ stands for the number of elements in the set M. That is, schools do not accept above their capacities.

For a student ϑ, $\mu^{-1}\left(\mu(\vartheta)\right) \setminus \{\vartheta\}$ represents the set of its peers under the matching μ.

Definition 4. *A matching μ is **individually rational** if for any student ϑ,*

$$u_\vartheta(\mu(\vartheta), \mu^{-1}\left(\mu(\vartheta)\right) \setminus \{\vartheta\}) \geq 0.$$

That is, no one prefers being unmatched to one's assignment under μ.

Definition 5. *A matching μ is **feasible** if for each student $\vartheta = (\theta, i)$ in each market i, $\mu(\vartheta) = (\ell, j)$, $j \neq i$ implies that $\{i \to j\} \in G$.*

Feasibility means that each student is matched to a school in a market into which one is allowed to switch.

Define a set of peers, which one gets after a deviation to a school v_ℓ^j under a matching μ as

$$\Theta(\ell, j; \mu) = \begin{cases} \mu^{-1}(\ell, j), & \text{if } |\mu^{-1}(\ell, j)| < q_\ell^j; \\ \mu^{-1}(\ell, j) \setminus \{\min(\mu^{-1}(\ell, j))\}, & \text{if } |\mu^{-1}(\ell, j)| = q_\ell^j. \end{cases}$$

Thus, if the school v_ℓ^j is full, and a student ϑ deviates to that school, ϑ pushes out a student with the lowest type.

Definition 6. *A feasible matching* μ *is* ***pairwise stable*** *(or simply stable) if it is individually rational and for any student* ϑ *of type* θ *and home market* i*:*

$$\text{if } u_\vartheta(\mu(\vartheta), \mu^{-1}(\mu(\vartheta)) \setminus \{\vartheta\}) < v_\ell^j + \alpha \cdot p(\Theta(\ell, j; \mu)) - c_{ij}, \text{ then}$$
$$|\mu^{-1}(\ell, j)| = q_\ell^j \text{ and } \theta \leq \min(\mu^{-1}(\ell, j)).$$

Pairwise stability means that for each student, all more preferred schools are filled up to capacity by students with higher types. Alternative stability definitions are discussed in Section 8.

In the following two sections we are going to, first, propose a sufficient condition on the graph of available market switches, G, which guarantees the existence of a stable matching. Then we will show that our condition is in a sense necessary, that is if G has cycles, then it is possible to find types, costs, and school values and capacities, such that no stable matching would exist.

4. Sufficiency

In this section we present a sufficient condition, which guarantees the existence of a stable matching. Under our condition there exists an algorithm, which produces a stable matching. Some properties of the algorithm and the stable outcome it produces are investigated below. We also compare our sufficiency condition with the pairwise alignment condition of Pycia (2012).

4.1. Construction of a stable matching.

The example presented in Section 2 illustrates that non-existence of stable matchings may come from the possibility of students cyclically switching their locations: a student X, born in market i_1, moves to market i_2 and pushes a student Y out of a school in his home market i_2, so that Y needs to switch to a different market. The student Y switches the market from i_2 to i_1, so that market i_1 becomes better, and X prefers to stay at home and not pay extra travelling costs. When X moves home, Y can go back, as his previous seat is now empty. Y returns to i_2 and we are back to the start of the cycle. This is summarised in Figure 3.

Similar pattern may arise with multiple market switches. E.g. if someone moves from a market i_1 to a market i_2 and pushes other student out, that other student moves from i_2 to i_3, and so on until a student is pushed from i_ℓ and moves to i_1. That makes i_1 attractive

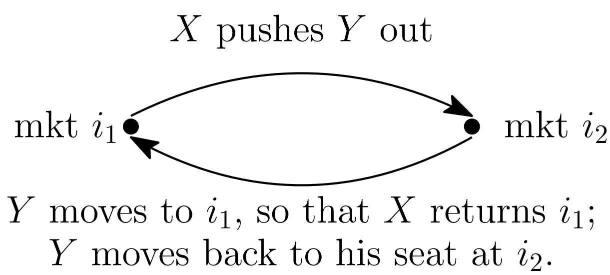

FIGURE 3. Cycle of length 2.

again, so that the first student returns, leaving an empty seat at i_2. Then the second student returns and so on.

Moreover, even non-directed cycles like $i_1 \to i_2 \to i_3$, $i_1 \to i_3$ can cause a problem: a student from i_1 goes to i_3, which is the most desired place for a student from i_2, so that the student from i_2 cannot go to i_3 and has to stay at i_2 (there are no more empty seats in i_3). However, when the student from i_2 stays at i_2, the student from i_1 may decide to stay with him/her at i_2, thus, leaving the seat at i_3 vacant. Therefore, the student from i_2 takes the empty seat and leaves the student from i_1 alone at i_2. So the student from i_1 switches back to i_3 and pushes the other student back to i_2, and we get a cyclical pattern, which prohibits the existence of a stable matching.

The following Theorem 1 proves that as long as no cycles exist in the switching graph G, a stable matching exists. Let us state this more formally.

Definition 7. *An* ***undirected graph*** *is a pair* (V, E)*, where* V *is a finite set and* $E \subset V \times V$*, such that for any* $i \in V$*,* $\{i \leftrightarrow i\} \notin E$ *and for any* $i, j \in V$ *if* $\{i \leftrightarrow j\} \in E$ *then* $\{j \leftrightarrow i\} \in E$*. Elements of* V *are called vertices and elements of* E *are called edges and are denoted by* $\{i \leftrightarrow j\}$*, where* $i, j \in V$*.*

Definition 8. *A* ***directed graph*** *is a pair* (V, E)*, where* V *is a finite set and* $E \subset V \times V$*, such that for any* $i \in V$*,* $\{i \to i\} \notin E$*. Elements of* V *are called vertices and elements of* E *are called directed edges and are denoted by* $\{i \to j\}$*, where* $i, j \in V$*.*

Definition 9. *Suppose* (V, E) *is a directed graph. Its* ***underlying undirected graph*** *is an undirected graph* (V, E_u)*, where* $E_u \subset V \times V$ *and* $\{i \leftrightarrow j\} \in E_u$ *if and only if* $\{i \to j\} \in E$ *or* $\{j \to i\} \in E$*.*

Definition 10. *A **tree** is an undirected graph that is connected and acyclic (contains no cycles).*

In other words, in a tree any two vertices are connected by exactly one path.

Definition 11. *An **oriented tree** is a directed graph such that, first, its underlying undirected graph is a tree and, second, if $\{i \to j\}$ belongs to the oriented tree, then $\{j \to i\}$ does not belong to it.*

In other words, the first condition precludes cycles of length 3 and more, and the second condition precludes directed cycles of length 2.

Definition 12. *An **oriented forest** is a disjoint union of finite number of oriented trees.*

In practice many markets are oriented forests. The most common structure is a star graph, where the center node represents a market available to everyone and peripheral nodes are some specific markets, available only to groups of people. For example, zoned public schools in the US, which accept only students who live in a district corresponding to a given school, represent those peripheral nodes. Simultaneously, magnet schools represent the central node of the star graph. Large fraction of them has a competitive admission process, does not require students to live in specific districts, and draws students from across the normal boundaries defined by authorities. For instance, eight out of nine specialized high schools in New York City accept students based on scores from the Specialized High School Admissions Test (NYC Department of Education (accessed October 1, 2019)). The qualifying score for each school changes year to year and depends upon the number of seats available and the scores of all the candidates. Seats are filled starting with the highest test scores. This fits nicely in our setting. Russian school system falls into the same category. There are public schools, which accept everyone who live nearby. They correspond to peripheral nodes of the star graph. Additionally, there are more competitive schools, to which everyone can apply, but has to pass the entrance exam. Those schools represent the center of the star.

Theorem 1. *Suppose that the switching graph G is an oriented forest. Then a stable matching exists. Such matching can be found by a finite iterative algorithm.*

The proof in the Appendix presents an algorithm that constructs a stable matching for any oriented forest. Note that the algorithm works not only for star-shaped graphs, but also for more complicated structures without cycles. Thus, as a market designer, one may experiment with very complicated structures and still be sure that a stable outcome would exist.

The algorithm consists of several iterations: in each iteration we fill some school up to its capacity and further remove this school and all its students from consideration. Let us say that this is a school ℓ in a market i. In each iteration, it is crucial that we manage to find an assignment of students to the school (ℓ, i) satisfying the following three properties. First, (ℓ, i) is the best school in its market i. Second, the students assigned to (ℓ, i) are of the highest types among all the students from $F_i \bigcup_{\{j \to i\} \in G} F_j$. Third, if the best student $\bar{\theta}_j$ from F_j got assigned to (ℓ, i), then for each other assignment to a school from $\{j\} \bigcup \{j' \mid \{j \to j'\} \in G\}$, which satisfies the first two properties, $\bar{\theta}_j$ should also be a part of this assignment, but his utility should be (weakly) less. That is, $\bar{\theta}_j$ should get a seat in the best schools in all markets, where he is allowed to switch, and should weakly prefer assignment to (ℓ, i).

It is non-trivial that such school exists at all, however, the acyclicity of G guarantees the existence. The stability of the matching constructed through such iterations follows from the observation that in the third property we only need to look at the highest student type in each market. Lower student types will agree to follow the highest one. See Appendix for the details of the proof. Example 2 in the Appendix applies the algorithm to the illustrative example from Section 2.

4.2. **Properties of the algorithm.** Let us discuss some properties of the algorithm and of the stable matching it produces.

First important property is that the choice of the starting market does not change the outcome of the algorithm. That is, it does not matter the best school in which market we try to fill the first. We show that in Appendix in Lemma 7.

Next, the matching which we get as the outcome of the algorithm from Theorem 1, has an assortative pattern: inside each market, students are allocated to schools in an assortative manner. That is, the better is a school in a market i, the higher types have students assigned to that school. Formally,

$$\forall i, \ell, \ell' \text{ s.t. } \ell < \ell' \text{ if } \theta \text{ is matched to } v^i_\ell \text{ and } \theta' \text{ is matched to } v^i_{\ell'}, \text{ then } \theta \geq \theta'.$$

Such construction serves as an instrument to make deviations inside a given market unprofitable.

Further, for a given oriented tree, at each round of the algorithm we take at most n steps (the worst is if we go from the root of the tree to a leaf covering all other $n-1$ markets). Then at each round we fill one school (including an outside option). Thus, in total we need at most $n(k_1+1+k_2+1+\ldots+k_n+1) = n\left(n+\sum_{i=1}^{n} k_i\right)$ units of time.

Let us remark that the presence of peer effects may preclude the existence of student-optimal or school-optimal matchings.[5] Consider, for example, the following model with one market and the average quality of peers as a peer effect function. Suppose that there are two schools each with capacity 2, $v_1^1 = 8$, $v_2^1 = 7$, and there are four students, $F_1 = \{10, 9, 5, 4\}$. Then there are two stable matchings: $(9, 10) \to v_1^1, (5, 4) \to v_2^1$ and $(9, 10) \to v_2^1, (5, 4) \to v_1^1$. Our algorithm picks the first one. Students disagree on which is better: 10 and 9 prefer the first matching, while 5 and 4 prefer the second one. The idea is that now having a high enough peer can force a student to stay at a lower ranked school. Thus, although 10 and 9 prefer v_1^1 per se, if they cannot coordinate on going to it, it is better for them to stay together at v_2^1.

Similarly, the presence of peer effects can make comparative statics ambiguous. In our case the only unambiguous change is a marginal increase in a quality of some school or a marginal increase in someone's type. This small increase is important, as it guarantees that an allocation per se does not change. In that case those who share an allocation at the improved school or with the improved peer gain, while others are not affected. However, when the increase is large enough to change the matching, we get high type students who are willing to switch to the improved school and those who are now pushed out and, thus, lose. Likewise, if we increase capacity at some school, someone else will be able to get into it and, thus, will benefit. Yet others are now suffering from additional low peer.

4.3. **Comparison with Pycia (2012).** The questions in Pycia (2012) are closely related to ours. The author investigates necessary and sufficient conditions for the existence of a core stable matching in a matching model with peer effects. His crucial condition is pairwise alignment of preferences. This means that if we fix two students and consider any two

[5]A student-optimal (school-optimal) matching is a stable matching which all students (schools) weakly prefer to any other stable matching.

assignments, under both of which those students share the same school, then they must agree on which assignment is better.[6] This requirement and ours are non-nested.

In one direction it is clear: if we consider a full graph on two vertices with very small switching costs c_{12}, c_{21}, very small α, and large enough values $\{v^i_\ell\}_{i,\ell}$ then the pairwise alignment is satisfied (costs and peer effects are negligible, so only schools qualities v^i_ℓ matter, and students agree on them). However, such graph has a cycle.

For another direction consider the following example. When there is only one market, $n = 1$, our algorithm leads to the assortative matching[7]. In that case students agree on which school is the best, and, thus, if we match the best students with the best school, there will be no reason to unilaterally deviate from such assignment. However, the case of only one market still can violate pairwise alignment condition for core stability of Pycia (2012).

The violation comes from the fact that different students in the same school can get different peer effects, as they have different sets of peers (a student θ is in the set of peers of a student θ', but not in the set of peers of oneself). For example, let $\alpha = 1$ and let "average peer" be the peer-effect function. Suppose we have two schools with values 10 and 9.5. The first school has capacity 3, while the second has capacity 2. We consider $\theta = 5$, $\theta' = 0$, and fill the remaining seat at the first school with additional zero. Then $u_\theta(10, \{0, 0\}) = 10 > 9.5 = u_\theta(9.5, \{0\})$, while $u_{\theta'}(10, \{5, 0\}) = 12.5 < 14.5 = u_{\theta'}(9.5, \{5\})$. Thus, θ and θ' disagree on which assignment is the best, and their preferences are not pairwise aligned.

In our setting, to satisfy pairwise alignment, peer-effects term should be almost constant when only one student is changed in the set of peers. That is, two students, who share the same school, should have almost identical peer-effects. This can be obtained by decreasing α or by flattening the peer-effect function $p(\cdot)$. However, if $\alpha > 0$ and $p(\cdot)$ is not a constant function, one can always find types of students and schools, so that the preferences are not pairwise aligned, as in the above example.

[6]Formally, in the language of our setting, preferences are pairwise aligned if for all students θ, θ', all schools s, s', and all sets of peers C, C', we have that $u_\theta(s, C \cup \{\theta'\}) \geq u_\theta(s', C' \cup \{\theta'\})$ if and only if $u_{\theta'}(s, C \cup \{\theta\}) \geq u_{\theta'}(s', C' \cup \{\theta\})$.

[7]We remark that the assortative matching does not have to be a unique equilibrium. For more details see Section 6.

In the situation with several markets, the pairwise alignment condition sometimes prohibits the appearance of certain specific cycles in the switching graph G, but this depends on the values of the switching costs. If two students θ and θ' are from different markets and there are two assignments to schools s and s' from distinct markets, then it must be that there is an undirected cycle: $\{\theta \to s\}, \{\theta' \to s\}, \{\theta' \to s'\}, \{\theta \to s'\}$, where we write student's or school's name instead of the name of the market one belongs to.[8] Then for any two given sets of peers C, C' one can choose switching costs such that $u_\theta(s, C \cup \{\theta'\}) > u_\theta(s', C' \cup \{\theta'\})$ and $u_{\theta'}(s, C \cup \{\theta\}) < u_{\theta'}(s', C' \cup \{\theta\})$, which violates pairwise alignment. Hence, if the switching costs satisfy these inequalities, then one of the edges of the above cycle should be prohibited in order for the pairwise alignment to hold.

We also remark that core stability investigated in Pycia (2012) is a more demanding condition than the pairwise stability, which is the focus of our paper. We discuss strengthening our stability notion in our setting in more detail in Section 8.

Another aspect in which our setting and setting of Pycia (2012) are non-nested is that formally the latter requires not just one given economy, but a rich enough set of economies. It is required that the set of possible preference profiles is large enough, and for stability all those profiles must be pairwise aligned. This richness together with pairwise alignment allows Pycia (2012) to ensure that there is a stable matching.

5. Necessity

In this section we show that if a directed graph G of available market switches has a cycle (not necessary directed), then there exists a set of parameters, for which there is no stable matching. Theorem 2, which is proved in the Appendix, summarizes the result.

Definition 13. *We say that an increasing function $f(x)$ of $x \geq 0$* ***grows slower than exponentially*** *if*

$$\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = 1.$$

For instance, the function x^k for any $k > 0$ grows slower than exponentially.

[8]The cycle can have 2, 3, or 4 edges depending on whether a student and a school are from the same market or not. I.e., if θ and s are from the same market, then the edge $\{\theta \to s\}$ disappears, and we have a cycle of length 3. If additionally θ' and s' are from the same market, then we are left with the cycle of length 2: $\{s \to s'\}, \{s' \to s\}$.

Theorem 2. *Assume that $p(\{x\})$ is strictly increasing but grows slower than exponentially as a function of $x \in \mathbb{R}_+$. If G has a cycle of length 2 or its underlying undirected graph has a cycle, then there exist values of $\{v^i_k\}_{i,k}$, $\{q^i_k\}_{i,k}$, $\{c_{ij}\}_{i,j}$, $\{F_i\}_i$ such that the resulting economy has no stable matching.*

Remark 1. *We need to assume that peer-effect function is not constant. Otherwise students do not care about their peers: they get the same constant utility from any set of peers. Thus we are left with a model without peer effects, where the classical Gale-Shapley algorithm produces a pairwise stable matching. The assumption that $p(\{x\})$ is strictly increasing as a function of $x \in \mathbb{R}_+$ helps us to get rid of the above.*

Remark 2. *Slow growth of function $p(\{x\})$ is a technical condition, which guarantees the existence of switching costs satisfying certain inequalities. We stick to slower than exponential growth to simplify the solution. We believe that Theorem 2 will also hold without imposing it.*

The construction in the proof is in the spirit of the illustrative example from the Section 2. We put the highest type M and the lowest type, 0, in the same home market, i. We choose costs such that it is always better to pay the cost and get a seat with a non-zero classmate than to stay with 0. Then if M goes to the foreign market j, he pushes someone away, and it eventually leads to some non-zero type going to the market i, so that M can go back to his best choice, i. If M stays at the market i, then positive types do not join M there, so that he is left with 0 as a peer, and deviates to the market j.

6. Uniqueness/non-uniqueness of a stable matching

In the previous sections we have seen that when G has no cycles, stable matchings exist. However, we have not explored whether there is only one stable matching or there are many of them. In this section we will answer the question of uniqueness/non-uniqueness of stable matchings for the two boundary cases: "no peer effects" ($\alpha = 0$) and "only peer effects" (α large enough).

We will show that when there are no peer effects, a stable matching can be found by applying Gale-Shapley algorithm (Gale and Shapley (1962)), and it is generically unique.[9] In contrast, when peer effects dominate, so that only one's classmates matter we get a multiplicity of equilibria.

6.1. **No peer effects.** We can calculate the equilibrium by iterative matching of the best schools and the most high-skilled students (that is, we apply student-proposing Gale-Shapley algorithm and break indifferences in an arbitrary way). When $\alpha = 0$ we get a special case of a model of Gale and Shapley (1962), where prohibition to go to a market can be interpreted as having a large negative utility from schools in that market. Thus, we are guaranteed the existence of a stable matching.

Proposition 3. *If $\alpha = 0$, then for any graph G there exists a stable matching in the above model.*

We can have more than one stable matching in two cases. First, if there are two or more students with the same type θ (possibly from different originating markets), so that a school does not know whom to accept for the last available seat. Second, if a student is indifferent between two schools, so that this student does not have exactly one best option to which to point in the student-proposing Gale-Shapley algorithm. That is, as long as, first, there do not exist students ϑ and ϑ' with types θ and θ' such that $\theta = \theta'$, and, second, there do not exist a triple of markets i, j, f and two schools (ℓ, i) and (k, j) such that $v_\ell^i - c_{fi} = v_k^j - c_{fj}$, the equilibrium is unique. The Lebesgue measure of the set

$$\left\{ \{v_\ell^i\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,k_i}}, \{c_{ij}\}_{i \neq j}, \{\theta_\ell^i\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,d_i}} : \theta_\ell^i = \theta_k^j \text{ or } v_\ell^i - c_{fi} = v_k^j - c_{fj} \text{ for some } f, (i,\ell) \neq (j,k) \right\}$$
$$\in \mathbb{R}^{\sum_{i=1}^n k_i} \times \mathbb{R}^{n(n-1)} \times \mathbb{R}^{\sum_{i=1}^n d_i}$$

is zero. So generically a stable matching is unique.

The intuition for uniqueness is that in the constructed matching everyone gets their best choice among schools which are not occupied by higher types. Thus, in any different matching someone will be worse off and will be able to deviate to one's allocation from the

[9]The Lebesgue measure zero set of student types, school values, and switching costs such that there are indifferences of the form $v_\ell^i - c_{fi} = v_k^j - c_{fj}$ or there are two students of the same type may lead to multiple stable matchings.

student-proposing Gale-Shapley algorithm. Proposition 4 summarizes uniqueness results and is proved in the Appendix.

Proposition 4. *If $\alpha = 0$, then for any graph G, any numbers of schools, their qualities and capacities, and any types of students per market the stable outcome of the above model is generically unique. That is, the Lebesgue measure of the subset of*

$$\left\{ \left\{ v_\ell^i \right\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,k_i}}, \left\{ c_{ij} \right\}_{i \neq j}, \left\{ \theta_\ell^i \right\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,d_i}} \right\} \in \mathbb{R}^{\sum_{i=1}^n k_i} \times \mathbb{R}^{n(n-1)} \times \mathbb{R}^{\sum_{i=1}^n d_i}$$

for which there are multiple stable matchings is zero.

6.2. **Only peer effects.** The next proposition illustrates that when α becomes large enough, so that peer effects dominate, and v's and c's become unimportant, the situation transforms to a coordination problem. High types would like to coordinate and stay together. They can coordinate on different schools. This gives us multiple equilibria. The idea is that instead of trying to put the best types in schools with the highest values, as is done in the algorithm of Theorem 1, we can put them, for example, in the schools with the second highest values, and they still will not deviate, as they are getting the highest possible peer effects.

Proposition 5. *If α is large enough, G has no cycles, and at least one market has at least* 2 *schools, then there are multiple stable matchings.*

7. Role of the assumptions and extensions

In this section we examine what role are various assumptions of the model playing, how important they are, and how generalizable they are. First, let us talk about the assumption on F_i. We impose that it has a large number of 0s. We use it to get rid of partially filled schools (completely empty schools do not cause a problem). The following example illustrates how a partially filled school can cause a problem for the existence of a stable matching even when the switching graph has no cycles.

Example 1. *("empty seats") Suppose that $\alpha = 1$ and there are* 2 *markets and* 1 *school per market, capacities are $q_1^1 = 2$, $q_1^2 = 3$, school values are $v_1^1 = 1$, $v_1^2 = 1$, student types are $F_1 = \{1, 10\}$, $F_2 = \{11\}$, the switching graph is $G = \{1 \to 2\}$, and the corresponding switching cost is $c_{12} = 7$. Thus, it is impossible to move from market* 2 *to market* 1*, and there are no cycles. However, there still does not exist a stable matching. Capacities are such* *that students can always be admitted to their home school, thus, no one will choose outside option* ($1 > 0$)*. Possible matchings are*

- *If* $(1, 10) \to v_1^1$*, then* 10 *deviates to* v_1^2*:*

$$u_{10}(1) = 1 + 1 = 2 < 1 + 11 - 7 = 5 = u_{10}(2);$$

- *If* $1 \to v_1^1$, $(10, 11) \to v_1^2$*, then* 1 *deviates to* v_1^2*:*

$$u_1(1) = 1 < 1 + 10.5 - 7 = 4.5 = u_1(2);$$

- *If* $10 \to v_1^1$, $(1, 11) \to v_1^2$*, then* 1 *deviates to* v_1^1*:*

$$u_1(1) = 1 + 10 = 11 > 1 + 11 - 7 = 5 = u_1(2);$$

- *If* $(1, 10, 11) \to v_1^2$*, then* 10 *deviates to* v_1^1*:*

$$u_{10}(1) = 1 > 1 + 6 - 7 = 0 = u_{10}(2).$$

Thus, there are no stable matchings in the above economy.

Zero types help to get rid of non-existence, because then if a student switches to some school, the peer effects in that school can only go up (someone with a lower type is pushed out). In contrast, with empty seats low types can switch and decrease the peer effects. In Example 1 this happened when 1 switched to the second market.

Example 1 illustrates that if there is a partially filled school in a market i and it is possible to switch from market j to i, then the existence of a stable matching may fail. However, we do not need to impose zero types in the markets, to which no one can switch (i.e. $\nexists j$ s.t. $\{j \to i\} \in G$). This is because in the algorithm in Theorem 1, when we compare different allocations and choose the most preferred one for the highest types, we ensure that if one does not want to go to a school ℓ in a market i, then one will not want to go to that school later (e.g. we cannot have a situation where 10 prefers to stay at home with 0 more than being abroad with 0 and 11, but after we fix such assignment, 10 wants to join 11 assuming 0 remains at home). If later the school will have empty seats, others may want to join (as 10 joins 11). However, if no one can switch to market i, by monotonicity inside markets of the algorithm, only low types will stay at the partially filled school ℓ, so that higher types do not have incentive to go back. When higher types were choosing whether to stay at home or not, they were considering even better peer-set at home, and still decided to go to a foreign school.

Note that if the peer-effect function is $p(\Theta) = \sup \Theta$, so that only the highest type matters, then we do not need zeros at all. The peer-effect function does not depend on how many zeros are in the set of peers:

$$p(\Theta) = \sup \Theta = \sup (\Theta \cup \{0\}) = p(\Theta \cup \{0\}), \; p(\emptyset) = 0 = p(\{0\}).$$

Thus, for such peer-effect function we do not need to impose "many zero types" assumption.

The second crucial assumption is that students inside any market have the same preferences, and students from different markets still agree on the relative order of schools in any given market. The former guarantees us that lower types do not deviate from an assignment as long as higher types of the same origin also stay. The latter guarantees that the highest types from different origins agree on the best school inside any market and, thus, if placed in that school, do not wish to deviate to a different school inside that market.

It is possible to relax the assumption of identical preferences of students from the same origin. We can instead assume that utility of a student with type θ who was born in a market i and attends a school v_ℓ^j with peers Θ is:

$$v_\ell^j + \alpha \cdot p(\Theta) - c_{ij} - c(\theta, i),$$

where for all $i, j, \ell, \Theta, \theta > \theta'$ if $v_\ell^j + \alpha \cdot p(\Theta \cup \theta') - c_{ij} - c(\theta, i) \geq 0$, then $v_\ell^j + \alpha \cdot p(\Theta \cup \theta) - c_{ij} - c(\theta', i) \geq 0$. This can be satisfied if, for example, $c(\theta, i)$ is an increasing function of θ, so that higher types also have higher costs. Alternatively, for discrete economy the above is equivalent to a finite set of equations for $c(\theta, i)$, and we can choose an arbitrary solution to those. Such generalization allows different students born in the same market to have different preferences. Yet, the relative utility between two different schools still remains the same. The proof of Theorem 1 is still valid in this setting. (We only need to add outside option as one more alternative to compare for each of the highest types, as now for large enough value of $c(\theta, i)$ a high type can have negative utility even from the best school and, thus, prefer to stay unmatched.)

The other way to generalize preferences is to go from linear function of v, c, and $p(\Theta)$ to an arbitrary function $u(v_k^j, c_{ij}, \Theta)$. Suppose that u is increasing in the first argument, v, decreasing in the second argument, c, and is a peer-effect function for any fixed v, c (see

Definition 2). Then for $\theta_i \leq \bar{\theta}_i$, if $\bar{\theta}_i$ does not deviate from v_k^j with a set of peers Θ to $v_{k'}^{j'}$ with a set of peers Θ', θ_i also stays at v_k^j:

$$u\left(v_k^j, \Theta \setminus \theta_i \cup \bar{\theta}_i, c_{ij}\right) \geq u\left(v_k^j, \Theta, c_{ij}\right) \geq u\left(v_{k'}^{j'}, \Theta', c_{ij'}\right).$$

Thus, Theorem 1 is still valid. Similarly, Theorem 2 if stated in terms of function u instead of p, remains valid.

Moreover, we can allow some dependence on one's own type for the peer effect function. That is, working with $\tilde{p}(\theta, \Theta) = p(\Theta) + f(\theta)$, where $f(\cdot)$ is non-negative and weakly decreasing still guarantees the existence of a stable matching. The interpretation of weakly decreasing $f(\theta)$ is that lower types need more guidance and, thus, benefit more from better sets of peers, while higher types are more independent and, thus, care less about their peers. For such peer-effect function the main idea of Theorem 1 still holds: if a high type prefers one school over the other, than so do low types ($f(\theta)$ cancels out when we make a comparison). The only difference is that now if a high type prefers to stay unmatched, lower types may still prefer to go to a school, as they have higher value of $f(\cdot)$. Thus, if a high type $\bar{\theta}_i$ prefers an outside option to all currently available schools, we cannot say that so do all students with type below $\bar{\theta}_i$, who are born in the same market i. For some $\theta' < \bar{\theta}_i$ born in market i we may get $v_\ell^j + \alpha(p(\Theta) + f(\theta')) - c_{ij} > 0 > v_\ell^j + \alpha(p(\Theta \setminus \{\bar{\theta}\} \cup \theta') + f(\bar{\theta})) - c_{ij}$. Thus, we need to leave all students with types below or equal to θ' born in market i in the mechanism for further steps. Those students may eventually be matched to a school, where they get positive utility.

Finally, let us analyze the switching graph G. Sometimes, as in the examples with religious or district schools, it is exogenously given. Yet there may be cases when there are no explicit restrictions on who can apply to a given set of schools. However, if for a group of schools, i, the utility associated with another group of schools, j, in the best possible matching (best school plus best peers) is less than switching costs from i to j (e.g. exams are too hard so that it is not worth an effort), then we can impose the condition that $\{i \to j\} \notin G$, which will not affect possible matchings. Such method allows us to construct a graph. Of course, if we want the graph to have no cycles, there should be a large set of prohibitively high costs.

8. Alternative definitions of stability

In this section we consider some modifications of the pairwise-stability notion that we use (Def. 6). We consider both weaker and stronger versions. We show that our algorithm from Theorem 1 with minor modifications still leads to stable outcomes under alternative stability definitions. We start by weakening the definition in the direction of fairness. The weaker versions allow students to either only push out weaker types and not take empty seats when deviating to a different school or, vice versa, only take an empty seat and not push anyone out. The former approach allows us to get rid of the assumption on large set of zero types in each market. Yet, it leads to multiple new equilibria, where schools are only partially filled. The more demanding, stronger notion, which we consider, is a group stability. It allows students to deviate to a different school jointly. In particular, we observe that the stronger is the notion of stability, the more zeros we need to guarantee the existence of a stable matching.

8.1. **Fairness.** There are two ways in which a matching can violate stability. There either can be a justified envy, where a student with the higher type wants to take a seat of a lower type student in a fully occupied school, or there can be waste in a sense that a school has an empty seat and there is a student, who wants to take that seat. Let us relax the stability notion by focusing on each of the two cases separately.

Definition 14. *A feasible matching μ is **envy-free** if it is individually rational and for any student ϑ of type θ and home market i:*

$$\text{if } u_{\vartheta}(\mu(\vartheta), \mu^{-1}(\mu(\vartheta)) \setminus \{\vartheta\}) < v_{\ell}^{j} + \alpha \cdot p(\mu^{-1}(\ell, j) \setminus \{\min(\mu^{-1}(\ell, j))\}) - c_{ij}, \text{ then } \theta \leq \min(\mu^{-1}(\ell, j)).$$

That is, a matching is envy-free if there are no two students such that the one with the higher type prefers taking the lower student's seat to his current assignment. Note that even when a school has an empty seat, a student is not allowed to take it and has to push someone else out. In this case, everyone in the class weakly benefits, as the peer-effect function is monotone and increases when we get rid of the lowest type. This differs from our stability definition (Def. 6), where taking an empty seat may decrease the peer effects. Thus, a stable matching with half-empty school can violate envy-freeness, if the peer-effect from a set of peers without the lowest type in a school is significantly greater than the peer effect from the same set of peers with the lowest type. However, the outcome of the algorithm from Theorem 1 is envy-free as all schools are either fully occupied, so one has to push the lowest type out to get a seat, or have only zero type students. If one switches to a school, where the only students are zero types, it does not matter whether one takes an empty seat or pushes some zero out, as the peer effect function of q and $q-1$ zeros is the same for $q>1$. If one switches to a school where the only classmate is one student with zero type ($q=1$), then pushing this student out is weakly worse. These claims follow from the increasing property of a peer effect function:

$$p(\emptyset) = 0 \leq p(\{0\}) = p(\{0,0\}) = p(\{0,\ldots,0\}).$$

There are many more envy-free matchings. For example, if we run a version of serial dictatorship and assign the highest type student in the economy to his most preferred school (the school with the highest value v_ℓ^i minus switching costs) and delete that pair, then take the second highest type student and assign him to his most preferred schools among the remaining ones, and so on, then we get an envy-free matching. Each school has at most one student, thus, there are no peer effects, and higher types cannot envy lower ones, as they chose their assignment earlier. Moreover, the empty matching, where everyone is assigned to the outside option also satisfies envy-freeness.

One can notice that the above degenerate examples of envy-free matchings do not require "many zeros" assumption. Furthermore, we can construct a matching with a bunch of fully occupied schools across markets.[10] One of the possible ways to do this is to run the algorithm from Theorem 1 and add a Pre-Step. We formally show the algorithm in Example 3 in the Appendix.

Let us now focus on the second component of pairwise stability, where students can only take empty seats but cannot push anyone out.

[10]Because we are dealing with a finite economy, where at least one envy-free matching exists, it is guaranteed that there also exists an envy-free matching which assigns the highest number of students to schools.

Definition 15. *A feasible matching μ has* ***no waste*** *if it is individually rational and for any student ϑ of type θ and home market i:*

$$\text{if } u_\vartheta(\mu(\vartheta), \mu^{-1}(\mu(\vartheta)) \setminus \{\vartheta\}) < v_\ell^j + \alpha \cdot p(\Theta(\ell, j; \mu)) - c_{ij}, \text{ then } |\mu^{-1}(\ell, j)| = q_\ell^j.$$

That is, a matching has no waste if there does not exist a student and a school with an empty seat such that the student prefers taking that empty seat to his current assignment. Pairwise stability requires no waste, thus, the matching obtained as an outcome of the algorithm from Theorem 1 has no waste. Yet, there are many other matchings, which have no waste, but are not pairwise stable.

For example, consider a market m such that there are no other market m' with $\{m' \to m\} \in G$, i.e. the market to which no outside student can be admitted. Such market exists if the switching graph G has no cycles. All seats in m are either empty or occupied by domestic students. Suppose that in the outcome of the algorithm from Theorem 1 at least two schools in m have non-zero types assigned to them. Then we can switch the assignment of $\vartheta' = (\theta', m)$ and $\vartheta'' = (\theta'', m)$ such that $\mu(\theta') = v_1^m$ and $\theta'' = \max\{\theta : \mu(\theta, m) = v_2^m\}$. That is, we alter the outcome of the algorithm and assign ϑ'' to v_1^m and ϑ' to v_2^m. Denote the altered assignment by μ'. All students from $\mu^{-1}(v_2^m)$ benefit from such change (ϑ'' gets both better school and better peers and others get a better peer). Students from $\mu^{-1}(v_1^m)$ now get lower utility, but they still do not want to switch to any school with empty seats. This is because μ is pairwise stable, so ϑ'' does not benefit from switching to any empty seat under μ, and utility of ϑ' under the new assignment is the same as utility of ϑ'' under old assignment:

$$u_{\vartheta'}(v_2^m, \mu^{-1}(v_2^m) \setminus \{\vartheta''\}) = u_{\vartheta''}(v_2^m, \mu^{-1}(v_2^m) \setminus \{\vartheta''\}).$$

Thus, ϑ' also does not benefit from switching to any empty seat under μ'. For any $\vartheta \in \mu^{-1}(v_1^m) \setminus \{\vartheta'\}$, the set of peers under μ' is still weakly better than $\mu^{-1}(v_2^m) \setminus \vartheta''$, as any student from the former set has weakly higher type than any student from the latter set. The lowest type in $(\mu')^{-1}(v_1^m)$ is θ'', which is weakly larger than anyone from $\mu^{-1}(v_2^m)$. Combining this with the fact that $v_1^m \geq v_2^m$, we get that for any $\vartheta \in \mu^{-1}(v_1^m) \setminus \{\vartheta'\}$,

$$u_\vartheta(v_1^m, (\mu')^{-1}(v_1^m) \setminus \{\vartheta\}) \geq u_{\vartheta''}(v_2^m, \mu^{-1}(v_2^m) \setminus \{\vartheta''\}),$$

and ϑ does not benefit from switching to any empty seat under μ'. Obviously, ϑ' envies ϑ'' under μ', and the above construction is no longer pairwise stable.

Note also that the "many zeros" assumption is still crucial for no wastefulness. Example 1 still applies: it does not have any non-wasteful matching.

8.2. **Group stability.** There are cases when we may expect students to communicate with each other and deviate to a different school jointly. Group stability accounts for such joint deviations.

Given a set of students C, define a set of peers which C gets after a joint deviation to a school (ℓ, j) under a matching μ as

$$\Theta(\ell, j, C; \mu) = \left\{ q_\ell^j \text{ highest types from } C \cup \mu^{-1}(\ell, j) \right\}.$$

Definition 16. *A feasible matching μ has a **blocking coalition** if there exists a school $s = (\ell, j)$ and a set of students C, such that*

$$\begin{cases} \forall \vartheta = (\theta, i) \in C \text{ either } i = j \text{ or } \{i \to j\} \in G, \\ C \cap \mu^{-1}(s) = \emptyset, \\ \min \mu^{-1}(s) < \max C, \\ u_\vartheta(\mu(\vartheta), \mu^{-1}(\mu(\vartheta)) \setminus \{\vartheta\}) < v_\ell^j + \alpha \cdot p(\Theta(\ell, j, C; \mu)) - c_{ij} \\ \qquad\qquad\qquad\qquad \forall \vartheta = (\theta, i) \in \Theta(\ell, j, C; \mu) \cap C. \end{cases}$$

The first two conditions guarantee that everyone from the blocking coalition is allowed to go to the school s, but they are currently assigned to different schools. The third condition means that at least one student from the coalition can successfully push someone out of the school s. The last condition ensures that everyone, who gets a seat after a joint deviation, benefits from the deviation. Note that only students from $\Theta(\ell, j, C; \mu) \cap C$ get seats at s after the deviation, so those are the only students who "block". Thus, if (C, s) is a blocking coalition, then so is $(\Theta(\ell, j, C; \mu) \cap C, s)$.

Definition 17. *A feasible matching μ is **group stable** if it is individually rational and it has no blocking coalition.*

Theorem 6. *Suppose that the switching graph G is an oriented forest and each market i has at least $\sum_{\ell=1}^{k_i} q_\ell^i$ zero types. Then a group stable matching exists. Such matching can be found by a finite iterative algorithm.*

The amount of zeros required for Theorem 6 illustrates a general pattern: the stronger is the notion of stability we consider, the more zeros we need.

The proof of Theorem 6 modifies the algorithm from Theorem 1. The main change to the algorithm from Theorem 1 is that in each market we now try to fill not the school with the highest value v, but the school which is most preferred by the highest type who can attend this market. That is, for each market m, we define the highest type who can attend this market as $\bar{\theta}(m) = \max\left\{F_m \bigcup_{\{m' \to m\} \in G} F_{m'}\right\}$. We then consider assignments to all schools in m, where a school is filled with the best possible students, that is assignment A^m_ℓ fills a school v^m_ℓ up to capacity with the best students from $F_m \bigcup_{m' \colon \{m' \to m\} \in G} F_{m'}$. Finally, we choose a school $(\ell^*(m), m)$ such that $\bar{\theta}(m)$ prefers $A^m_{\ell^*(m)}$ among all $\{A^m_\ell\}_{\ell=1}^{k_m}$. Then we use schools $\left\{v^m_{\ell^*(m)}\right\}_m$ instead of $\{v^m_1\}_m$ in the algorithm of Theorem 1.

Remark 3. *In practice we need fewer than* $\sum_{\ell=1}^{k_i} q^i_\ell$ *zeros. In each market, after the algorithm assigns all non-zero types to some schools, we need to fill the remaining schools and seats with zero types. Thus, if the difference between the total number of seats and the total number of non-zero type is small, only few zeros will actually be assigned to a school. Most of zeros will not be matched to a school and will be assigned to the outside option.*

The existence of a group stable matching provides additional insights into the multiplicity of pairwise stable matchings. Because a group stable matching is also a pairwise stable, we know that if in the process of running the algorithm from Theorem 6 at some moment a school $\left\{v^m_{\ell^*(m)}\right\}_m \neq v^m_1$ is filled, then there are multiple pairwise stable matchings. (Then the group stable matching found in Theorem 6 differs from the pairwise stable one found in Theorem 1.)

9. Conclusion

When we think about many real life examples (e.g. school/college/internship/etc. matchings), peer effects should be a necessary component of students preferences. Thus, it seems crucial to be able to identify conditions for the existence of a stable matching in the presence of peer effects. Moreover, it is worth being able to explicitly construct a stable matching.

Current paper provides an algorithm, which can be used to construct a stable matching in the presence of peer effects. The sufficient (and in some sense necessary for the existence of a stable matching) condition for the algorithm to work is that the graph, which governs the ability of students to apply to different schools, does not have cycles (nor directed, nor undirected). The algorithm uses school values and capacities, students' types and their costs associated with applying to different schools, and a peer effect function as inputs. The algorithm takes a finite amount of time, which is polynomial in the number of schools. Therefore, it is possible for a central planner to implement such mechanism if one has enough information regarding the underlying economy. The resulting stable matching is assortative inside each market (but not across markets). Duflo et al. (2008) show by means of a randomized experiment in Kenya that students benefit from tracking (that is from being assigned to classes assortatively). Thus, we can view the assortative pattern of the outcome of our algorithm as an advantage.

In countries like Russia or China, where government has a power to prohibit groups of people to applying to some schools, the existence result of our paper can be used in market design. Armed with the knowledge that with cycles there may be no stable matching, a market designer can separate schools and students into groups without cycles to ensure the existence of a stable matching. Moreover, one can use the existence result to analyse outcomes of various policy changes. Based on the graph structure before and after the change, one can argue whether to expect stability or not. For example, before Brexit both EU and UK citizens were paying the same tuition while studying in the UK. That is, they were members of the same "market". However, it is expected that after Brexit the tuition will increase for EU citizens studying in the UK, which will separate one market into two different markets with switching costs corresponding to the increase in tuition levels.

In case of decentralized markets, we may view our stable matching as an outcome of a decentralized game between schools and students. As is common for an equilibrium notions, we may get multiple stable matchings. In particular, we do not have a unique stable matching when a peer effect component is very important (i.e. α is large enough). When α is large enough, our model resembles a coordination problem, which is known to have multiple equilibria. In contract, when peer effects are negligible (i.e. $\alpha \approx 0$) we go back to a classical many-to-one matching problem with identical preferences on the schools side, which has a unique solution.

Our algorithm and existence condition rely on the structure of the switching graph. It is still an open question whether we can formulate additional conditions to ensure existence, if we do not have the graph as exogenously given, but start from costs per se. Obviously, we know that if a cost of going from i to j is more than utility from the best outcome in j, then we can erase an edge $\{i \to j\}$. Yet, it may be possible to say something more for intermediate values of costs based on their relative values when compared to feasible utilities even in the presence of cycles. Which intermediate values would guarantee the existence of a stable matching in the presence of cycles? This issue is left for further research.

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10. Appendix

Proof of Theorem 1. Let us provide an iterative construction, which leads to a stable matching in a finite time. We work separately with each tree from the forest. Fix any tree from the forest and denote it G. Let G_u be its underlying undirected graph.

Choose an arbitrary node (vertex) to be the root of the tree and denote it as m_0. We divide vertices into groups, which we call "levels". The root belongs to level 0. Level k consists of vertices which have the shortest undirected path to m_0 of length k in G_u (they are connected to m_0 by the path consisting of k edges). If a vertex from level k is connected by an edge with a vertex from level $k+1$, we call the latter a "child" of the former, and the former a "parent" of the latter. A vertex with no children is called a terminal node or a leaf. Assume that non-empty levels are $\{0, 1, \ldots, K\}$.

Vertices from level k are denoted $m_{k,1}, m_{k,2}, \ldots$. The notation is illustrated in Figure 4, where $m_{3,1}, m_{3,2}, m_{3,3}, m_{2,2}$, and $m_{2,3}$ are terminal nodes and $K = 3$. Note that the above notation does not depend on the directed structure of the graph.

To account for the directed structure of the graph, we introduce two more notations. Whenever a vertex i is a child of a vertex j we say that i is a directed child of j if $\{j \to i\}$ belongs to the switching graph G and we say that i is an anti-directed child of j if $\{i \to j\}$ belongs to the switching graph G. For example, in Figure 4, $m_{1,1}$ is a directed child of m_0, while $m_{1,2}$ is an anti-directed child of m_0.

In the following procedure we treat the outside option as the worst school with fixed zero utility. In each step of the algorithm we tentatively fill some school up to its capacity with the best students, who can be at the school's market. For each market, whose students got seats in the school in the tentative assignment, we take the best representative (student with the highest type from that market). The goal is to find a tentative assignment, where all "best representatives" do not want to deviate to a different tentative assignment constructed

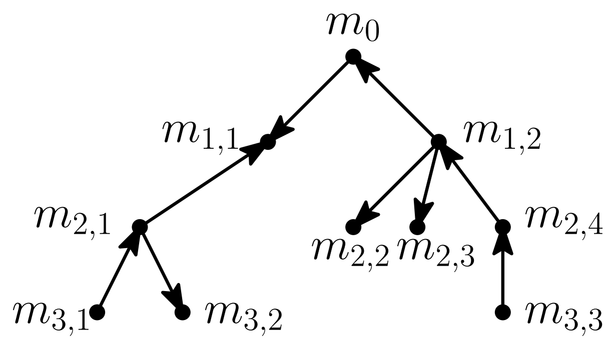

FIGURE 4. Directed tree: example.

in the same way by filling some school up to capacity with the highest types, who can be at the school's market. This allows us to fix this assignment and to restart the algorithm with the smaller economy.

In our search, we sequentially consider markets of growing levels. The key property is that when we reach level k, the markets from smaller levels are no longer relevant for the search.

We proceed to the detailed description.

Algorithm:

Step 0**:** Consider the following tentative assignment to the best school at m_0 (school $v_1^{m_0}$). Allocate all the best students[11] from all the markets, which are connected to m_0 by a directed edge (directed towards m_0), up to capacity to the school $v_1^{m_0}$. Note that since, by our assumptions, F_i has a large number of zero types, we are always able to fill the school $v_1^{m_0}$ up to capacity if there is at least one non-zero type assigned to that school. (If we cannot fill the school up to capacity, then only zeros occupy non-empty seats.)

Consider alternative assignments, constructed in the same way as the tentative one: fill some school up to capacity with the best eligible students. At the current step the only relevant alternative assignments are to the best schools in level 1 and 2 markets, $v_1^{m_{1,z}}$ and $v_1^{m_{2,z'}}$ for various z, z'.

[11]In the algorithm, when there are more students of the same type then available seats, resolve indifferences in favor of domestic students.

Denote by $\bar{\theta}_{k,z}$ the best student from the market $m_{k,z}$. For each anti-directed child of m_0 denoted by $m_{1,z}$ and such that $\bar{\theta}_{1,z}$ gets a seat at $v_1^{m_0}$ in the tentative assignment, ask $\bar{\theta}_{1,z}$ what assignment he/she prefers the most among the tentative one and alternative ones at $v_1^{m_{1,z}}$ and at $v_1^{m_{2,z'}}$ for all $m_{2,z'}$ such that $\{m_{1,z} \to m_{2,z'}\} \in G$. That is, whether one prefers the above allocation at $v_1^{m_0}$, or the allocation, where we put top students from $m_{1,z}$ and its anti-directed children markets to the best school at $m_{1,z}$, or the allocation, where we put top students from $m_{1,z}$ along with all other eligible markets to $m_{1,z}$'s directed child market $m_{2,z'}$. Similarly if the best student from m_0, $\bar{\theta}_0$, gets a seat at $v_1^{m_0}$, then we ask what assignment he/she prefers the most among the tentative one and alternative ones at $v_1^{m_{1,z''}}$ for all markets $m_{1,z''}$ where one can move from m_0 (all directed children markets).

Depending on the structure of the assignments and on the preferences of the students identified above, we proceed as follows:

0.1: If $\bar{\theta}_0$ gets a seat at $v_1^{m_0}$ in the tentative assignment and does not get a seat at some $v_1^{m_{1,z}}$ for a directed child $m_{1,z}$ of m_0 in an alternative assignment, move to Step 1 with the market $m_{1,z}$;

0.2: If some $\bar{\theta}_{1,z}$ gets a seat at $v_1^{m_0}$ in the tentative assignment and does not get a seat at $v_1^{m_{1,z}}$ in an alternative assignment, move to Step 1 with the market $m_{1,z}$;

0.3: If some $\bar{\theta}_{1,z}$ gets a seat at $v_1^{m_0}$ in the tentative assignment and does not get a seat at $v_1^{m_{2,z'}}$ in an alternative assignment, where $\{m_{1,z} \to m_{2,z'}\} \in G$, move to Step 2 with the market $m_{2,z'}$;

0.4: If $\bar{\theta}_0$ gets a seat in the tentative assignment and all alternative assignments in directed child markets of m_0 and prefers some $v_1^{m_{1,z''}}$, move to Step 1 with the market $m_{1,z''}$;

0.5: If for some anti-directed child $m_{1,z}$ of m_0, $\bar{\theta}_{1,z}$ gets a seat in the tentative assignment and in all alternative assignments at $m_{1,z}$ and its directed children markets and prefers the alternative assignment at the market $m_{1,z}$, move to Step 1 with the market $m_{1,z}$;

0.6: If for some anti-directed child $m_{1,z}$ of m_0, $\bar{\theta}_{1,z}$ gets a seat in the tentative assignment and in all alternative assignments at $m_{1,z}$ and its directed children markets and prefers an alternative assignment at the market $m_{2,z'}$, move to Step 2 with the market $m_{2,z'}$;

0.7**:** Otherwise fix the above tentative assignment at $v_1^{m_0}$. Delete that school and its students. Go back to Step 0 with the new economy.

Remark: *In the economy of Figure 4 that would mean asking $\bar{\theta}_0$ and $\bar{\theta}_{1,2}$. We ask $\bar{\theta}_0$ whether he/she prefers seating at the market m_0 with the students from m_0 and $m_{1,2}$ or seating at the market $m_{1,1}$ with the students from m_0, $m_{1,1}$, and $m_{2,1}$. We ask $\bar{\theta}_{1,2}$ whether he/she prefers seating at the market m_0 with the students from m_0 and $m_{1,2}$, or seating at the market $m_{1,2}$ with the students from $m_{1,2}$ and $m_{2,4}$, or seating at the market $m_{2,2}$ with the students from $m_{1,2}$, and $m_{2,2}$, or seating at $m_{2,3}$ with the students from $m_{1,2}$, and $m_{2,3}$. In each case, if the answer cannot be obtained due to schools, where alternative assignments do not include the asked student, we move to Step 1 or 2. Similarly, if the asked student prefers an alternative assignment to the tentative one, we move to Step 1 or 2.*

Step 1: Consider the market identified at Step 0 (the market, which triggered "move to Step 1" action). Denote it $m_{1,z}$.

The new tentative assignment is as follows. Allocate the best students from the market $m_{1,z}$, its parent m_0, and its anti-directed children markets to the best school at $m_{1,z}$ up to capacity. Alternative assignments are constructed in the same way: we fill a school up to capacity with the best eligible students. For each anti-directed child of $m_{1,z}$ denoted by $m_{2,z'}$ and such that $\bar{\theta}_{2,z'}$ gets a seat at $v_1^{m_{1,z}}$ in the tentative assignment, ask what assignment he/she prefers the most among the tentative assignment at $v_1^{m_{1,z}}$ and alternative assignments at $v_1^{m_{2,z'}}$ and at $v_1^{m_{3,z''}}$ for all $m_{3,z''}$ such that $\{m_{2,z'} \to m_{3,z''}\} \in G$. Similarly if the best student from $m_{1,z}$, $\bar{\theta}_{1,z}$, gets a seat at $v_1^{m_{1,z}}$ then we ask what assignment he/she prefers the most among the tentative one at $v_1^{m_{1,z}}$ and alternative ones at $v_1^{m_{2,z''}}$ for all markets $m_{2,z''}$ where one can move from $m_{1,z}$ (all directed children markets).

Note that we do not need to ask $\bar{\theta}_0$ even if one gets a seat at the tentative assignment at $v_1^{m_{1,z}}$. If $\bar{\theta}_0$ gets a seat at $v_1^{m_{1,z}}$ and we get that market from the previous step, then it was $\bar{\theta}_0$'s first choice. Similarly, we do not need to ask $\bar{\theta}_{1,z}$ whether he/she would prefer an alternative assignment at m_0. If it is possible to travel from $m_{1,z}$

to m_0 and we get $m_{1,z}$ from Step 1, it means either $\bar{\theta}_{1,z}$ does not get a seat in the tentative assignment at $v_1^{m_{1,z}}$, so we do not ask $\bar{\theta}_{1,z}$ at all, or the current tentative assignment at $v_1^{m_{1,z}}$ is $\bar{\theta}_{1,z}$'s first choice, thus, it is preferred to m_0 by $\bar{\theta}_{1,z}$.

Depending on the structure of the assignments and on the preferences of the students identified above, we proceed as follows:

- **1.1:** If $\bar{\theta}_{1,z}$ gets a seat at $v_1^{m_{1,z}}$ in the tentative assignment and does not get a seat at some $v_1^{m_{2,z'}}$ for a directed child $m_{2,z'}$ of $m_{1,z}$ in an alternative assignment, move to Step 2 with the market $m_{2,z'}$;
- **1.2:** If some $\bar{\theta}_{2,z'}$ gets a seat at $v_1^{m_{1,z}}$ in the tentative assignment and does not get a seat at $v_1^{m_{2,z'}}$ in an alternative assignment, move to Step 2 with the market $m_{2,z'}$;
- **1.3:** If some $\bar{\theta}_{2,z'}$ gets a seat at $v_1^{m_{1,z}}$ in the tentative assignment and does not get a seat at $v_1^{m_{3,z^*}}$ in an alternative assignment, where $\{m_{2,z'} \to m_{3,z^*}\} \in G$, move to Step 3 with the market m_{3,z^*};
- **1.4:** If $\bar{\theta}_{1,z}$ gets a seat in the tentative assignment and all alternative assignments in directed child markets of $m_{1,z}$ and prefers some $v_1^{m_{2,z''}}$, move to Step 2 with the market $m_{2,z''}$;
- **1.5:** If for some anti-directed child $m_{2,z'}$ of $m_{1,z}$, $\bar{\theta}_{2,z'}$ gets a seat in the tentative assignment and in all alternative assignments at $m_{2,z'}$ and its directed children markets and prefers the alternative assignment at the market $m_{2,z'}$, move to Step 2 with the market $m_{2,z'}$;
- **1.6:** If for some anti-directed child $m_{2,z'}$ of $m_{1,z}$, $\bar{\theta}_{2,z'}$ gets a seat in the tentative assignment and in all alternative assignments at $m_{2,z'}$ and its directed children markets and prefers an alternative assignment at some market m_{3,z^*}, move to Step 3 with the market m_{3,z^*};
- **1.7:** Otherwise fix the above tentative assignment at $v_1^{m_{1,z}}$. Delete that school and its students. Go back to Step 0 with the new economy.

...

Step k: Do the same thing as in the previous steps, but with the best school at market $m_{k,z}$, $v_1^{m_{k,z}}$. It is the market, which triggered transition to Step k in previous steps (either at Step $k-1$ or at Step $k-2$).

The new tentative assignment is as follows. Allocate the best students from the market $m_{k,z}$, its parental market, and its anti-directed children markets to the best school at $m_{k,z}$ up to capacity. For each anti-directed child of $m_{k,z}$ denoted by $m_{k+1,z'}$ and such that $\bar{\theta}_{k+1,z'}$ gets a seat at $v_1^{m_{k,z}}$ in the tentative assignment, ask $\bar{\theta}_{k+1,z'}$ what assignment he/she prefers the most among the tentative assignment at $v_1^{m_{k,z}}$ and the alternative assignments at $v_1^{m_{k+1,z'}}$ and at $v_1^{m_{k+2,z''}}$ for all $m_{k+2,z''}$ such that $\{m_{k+1,z'} \to m_{k+2,z''}\} \in G$. Similarly if the best student from $m_{k,z}$, $\bar{\theta}_{k,z}$, gets a seat at $v_1^{m_{k,z}}$, then we ask what assignment he/she prefers the most among the tentative one at $v_1^{m_{k,z}}$ and alternative ones at $v_1^{m_{k+1,z''}}$ for all markets $m_{k+1,z''}$ where one can move from $m_{k,z}$ (all directed children markets).

As before, we do not need to ask students from the parental market of $\bar{\theta}_{k,z}$ even if one gets a seat at $v_1^{m_{k,z}}$. If the best type from the parental market gets a seat at $v_1^{m_{k,z}}$, and we get the market $m_{k,z}$ from Step $k-1$ or $k-2$, then $v_1^{m_{k,z}}$ was the first choice for the best student from $m_{k,z}$'s parental market. Similarly, we do not need to ask $\bar{\theta}_{k,z}$ about the alternative assignment at its parental market. If it is possible to travel from $m_{1,z}$ to the parental market and we get $m_{k,z}$ from previous steps, then $m_{1,z}$ is an anti-directed child and then we must get this market from Step $k-1$ (because $m_{k,z}$ is an anti-directed child, it does not participate in Step $k-2$). Thus, either $\bar{\theta}_{k,z}$ does not get a seat in the tentative assignment at $v_1^{m_{k,z}}$, so we do not ask $\bar{\theta}_{k,z}$ at all, or the current tentative assignment is $\bar{\theta}_{k,z}$'s first choice, thus, it is preferred to the alternative assignment in the parental market.

Depending on the structure of the assignments and on the preferences of the students identified above, we proceed as follows:

k.1: If $\bar{\theta}_{k,z}$ gets a seat at $v_1^{m_{k,z}}$ in the tentative assignment and does not get a seat at some $v_1^{m_{k+1,z'}}$ for a directed child $m_{k+1,z'}$ of $m_{k,z}$ in an alternative assignment, move to Step $k+1$ with the market $m_{k+1,z'}$;

k.2: If some $\bar{\theta}_{k+1,z'}$ gets a seat at $v_1^{m_{k,z}}$ in the tentative assignment and does not get a seat at $v_1^{m_{k+1,z'}}$ in an alternative assignment, move to Step $k+1$ with the market $m_{k+1,z'}$;

k.3**:** If some $\bar{\theta}_{k+1,z'}$ gets a seat at $v_1^{m_{k,z}}$ in the tentative assignment and does not get a seat at $v_1^{m_{k+2,z^*}}$ in an alternative assignment, where $\{m_{k+1,z'} \to m_{k+2,z^*}\} \in G$, move to Step $k+2$ with the market m_{k+2,z^*};

k.4**:** If $\bar{\theta}_{k,z}$ gets a seat in the tentative assignment and all alternative assignments in directed child markets of $m_{k,z}$ and prefers some $v_1^{m_{k+1,z''}}$, move to Step $k+1$ with the market $m_{k+1,z''}$;

k.5**:** If for some anti-directed child $m_{k+1,z'}$ of $m_{k,z}$, $\bar{\theta}_{k+1,z'}$ gets a seat in the tentative assignment and in all alternative assignments at $m_{k+1,z'}$ and its directed children markets and prefers the alternative assignment at the market $m_{k+1,z'}$, move to Step $k+1$ with the market $m_{k+1,z'}$;

k.6**:** If for some anti-directed child $m_{k+1,z'}$ of $m_{k,z}$, $\bar{\theta}_{k+1,z'}$ gets a seat in the tentative assignment and in all alternative assignments at $m_{k+1,z'}$ and its directed children markets and prefers an alternative assignment at some market m_{k+2,z^*}, move to Step $k+2$ with the market m_{k+2,z^*};

k.7**:** Otherwise fix the above tentative assignment at $v_1^{m_{k,z}}$. Delete that school and its students. Go back to Step 0 with the new economy.

...

Step K**:** We must stop if we have reached a node $m_{K,z}$, as by definition it is a terminal node. Only students from $m_{K,z}$ and from its parental markets can get a seat at $v_1^{m_{K,z}}$. Thus, all "If..." clauses in the above list from the Step k are not satisfied. We are left with the last option "Otherwise fix the above assignment...", i.e. we finalize the assignment and return to Step 0 with the new smaller economy.

In the above algorithm, once all students and all schools in a given market are removed from the economy, we remove this market from the consideration and continue independently in each connected component of the remaining graph.

Let us explain why the above algorithm leads to a stable matching. Note that in each step we are trying to get the best possible scenario for the highest type in some market. Thus, that type does not want to deviate inside the market: schools in a given market by construction have decreasing peer effect and value, thus, there is no reason to deviate to a school with a larger number in the same market. Here we are using the properties of a peer effect function, which imply that if peers in one set are weakly larger than in the other, then the former set has weakly higher value of a peer effect function. Moreover, there is no reason to deviate to any other feasible market, as in the algorithm we were choosing the best market.

We also need to show that students, which are assigned to some school during some step in the algorithm and were not the highest types in that step, still do not want to deviate. Suppose we implement an assignment at Step k, that is, we fill a school at some market $m_{k,z}$. Thus, if the highest type from market $m_{k,z}$, $\bar{\theta}_{k,z}$, the highest type from $m_{k,z}$'s parental market, $\bar{\theta}_{k-1,z'}$, and any of $m_{k,z}$'s children markets highest types, $\bar{\theta}_{k+1,z''}$, get a seat at $v_1^{m_{k,z}}$, then it is their desired allocation. (They get a seat in all of the markets, where they are eligible to travel, but choose $m_{k,z}$.) Let us look at the second highest type from the market $m_{k,z}$, θ', and show that θ' does not deviate. If θ' stays at the same school as $\bar{\theta}_{k,z}$, θ' gets higher utility, as its set of peers is better:

$$peers(\theta') = peers(\bar{\theta}_{k,z}) \cup \{\bar{\theta}_{k,z}\} \setminus \{\theta'\}.$$

Moreover, deviating to a different market leads to a weakly lower utility than $\bar{\theta}_{k,z}$ was getting, while we were doing a comparison at Step k (or $k-1$ or $k-2$). Deviating to the best school at another market means sharing weakly worse set of peers than $\bar{\theta}_{k,z}$ had: $\bar{\theta}_{k,z}$ is no longer there and is replaced by someone worse. Moreover, if θ' was at that school with $\bar{\theta}_{k,z}$, then $\bar{\theta}_{k,z}$ is replaced by someone weakly worse than θ'. The best possible set of peers is, thus, $peers(\bar{\theta}_{k,z})$ from that school at the moment of comparison at Step k (or $k-1$ or $k-2$). If θ' was not at that school with $\bar{\theta}_{k,z}$, then he takes $\bar{\theta}_{k,z}$'s place and, again, gets peers no better than $\bar{\theta}_{k,z}$ had. Thus, deviating to a different market leads to a weakly smaller utility than $\bar{\theta}_{k,z}$ had at that market, while staying with $\bar{\theta}_{k,z}$ leads to a weakly higher utility than $\bar{\theta}_{k,z}$ has. Thus, second highest type from $m_{k,z}$ does not deviate. Applying induction argument, other students from $m_{k,z}$, $m_{k-1,z'}$, and $m_{k+1,z''}$ also do not deviate. □

Example 2. *Let us consider the economy described in Section 2. It consists of two schools, $\mathcal{A}$ and $\mathcal{B}$, and four student with types* $0, 7, 8, 10$. *As in the second part of the illustrative example, let us assume that student* 10 *is not allowed to apply to school $\mathcal{B}$. That is, there are two markets, one with the school $\mathcal{A}$ and the set of students* $\{0, 10\}$ *(say, market* 1*) and another with the school $\mathcal{B}$ and the set of students* $\{7, 8\}$ *(say, market* 2*). Thus, the switching* *graph is* $G = \{2 \to 1\}$*. School values are* $v_{\mathcal{A}} = 10$*,* $v_{\mathcal{B}} = 9.5$ *and the switching cost*[12] *is* $c_{21} = 4$.

Suppose that we start the algorithm at the market 1 *and try to fill the school* $\mathcal{A}$*. We tentatively assign the highest feasible types to the school* $\mathcal{A}$*. That is, we assign* 10 *and* 8 *to* $\mathcal{A}$*. The highest type from the market* 1*, student* 10*, cannot switch to any other market, so this type cannot deviate from the tentative assignment. The highest type from the market* 2*, student* 8 *can deviate to its home-school,* $\mathcal{B}$*. Thus, we need to check whether* 8*, prefers current tentative assignment to the alternative assignment which puts all top eligible students to the school* $\mathcal{B}$*. The latter assignment puts the highest eligible types,* 8 *and* 7*, to the school* $\mathcal{B}$ *(*10 *cannot go to the market* 2*).*

$$u_8(\mathcal{A}, 10) = 10 + 10 - 4 = 16 < 9.5 + 7 = 16.5 = u_8(\mathcal{B}, 7).$$

Thus, 8 *prefers the alternative assignment, the tentative assignment is not feasible, and we move to the next market. We consider a tentative assignment at the market* 2 *and fill the school* $\mathcal{B}$ *with the best students. That is, we now tentatively assign* 7 *and* 8 *to* $\mathcal{B}$ *(*10 *cannot go to the market* 2*). We need to check that the highest types from each market, who are assigned seats in the tentative assignment, do not want to deviate to an alternative assignment. That is, we need to check that* 8 *prefers the tentative assignment at* $\mathcal{B}$ *to the alternative assignment, where* 8 *shares a seat with* 10 *in the school* $\mathcal{A}$*. As we have checked,* $u_8(\mathcal{B}, 7) > u_8(\mathcal{A}, 10)$*, so* 8 *does not want to deviate and we finalize the current tentative assignment at* $\mathcal{B}$*. We are left with the school* $\mathcal{A}$ *and the students* $0, 10$*. So we assign* 10 *and* 0 *to the school* $\mathcal{A}$*. They cannot switch to different markets, so we finalize this assignment, and we are left with the stable outcome* $\{10, 0\} \to \mathcal{A}$*,* $\{8, 7\} \to \mathcal{B}$*.*

Note, that if we started from tentatively filling market 2*'s best school, the outcome would not change. This is related to the property of the algorithm, which says that the choice of the initial market does not matter. (See Lemma 7.)*

Lemma 7. *The outcome of the algorithm in Theorem 1 does not depend on the choice of the starting market (root of the tree* m_0*).*

[12]To get the switching cost subtract the utility the student 8 gets at the school $\mathcal{A}$ without peers from the utility the student 10 gets at the school $\mathcal{A}$ without peers: $c_{21} = v_{10}(\mathcal{A}) - v_8(\mathcal{A}) = 10 - 6 = 4$.

Proof. Suppose we start from some market m^* and first fill a school v^i_ℓ with highest types from markets $\{m_1, \ldots, m_r\}$. Let us explain that if we start with a different market, v^i_ℓ will still be filled with the same students. Because we assigned highest types from markets $\{m_1, \ldots, m_r\}$ to v^i_ℓ at the first step of the algorithm, when starting from m^*, it must be that school v^i_ℓ filled with the best possible peers is the best option for $\bar{\theta}_{m_1}, \ldots, \bar{\theta}_{m_r}$, where $\bar{\theta}_{m_j}$ denotes the maximal type in F_{m_j}. Thus, when we start from a different market, we cannot assign any of $\bar{\theta}_{m_1}, \ldots, \bar{\theta}_{m_r}$ to another school, as they will be asked to compare that other option with v^i_ℓ filled with the best possible peers, and they will choose the latter. Similarly, we cannot fill v^i_ℓ with other students, as until we assign $\bar{\theta}_{m_1}, \ldots, \bar{\theta}_{m_r}$ somewhere, they represent a subset of peers admitted to v^i_ℓ. The same reasoning applies to schools filled later in the algorithm, which starts at m^*. □

Proof of Theorem 2. Suppose G has a cycle. Choose the smallest cycle of G. Without loss of generality let us assume that it involves markets $1, \ldots, \ell$.

We now specify parameters (number of schools per market, their qualities and capacities, switching costs between markets, and types of students). Let us fix $k_i = 1$, $q^i_1 = 2$ for all markets i. That is, there is only one school per market with capacity 2.

The further procedure splits into two parts: we separately make a specification for the cycle markets $1, \ldots, \ell$ and for the outside markets $\ell+1, \ldots, n$. First, let us show that for any choices for the cycle markets, we can make complimentary choices for the outside markets so that the matching problem splits into two independent ones: for the cycle markets and for the rest.

Indeed, choose all switching costs c_{ij} with either $i > \ell$ or $j > \ell$ in an arbitrary way. Further, set for $i > \ell$, $F_i = \{M_i, M_i\}$, where M_i is an increasing sequence of i, such that $M_{\ell+1}$ is strictly larger than types of all students in markets $1, \ldots, \ell$. That is, the higher is the number of a market, the better students occupy it, and all the students in the outside markets are better than those in the cycle markets. Next, for $i > \ell$, we set v^i_1 to be an increasing function of i such that for each $i > j > \ell$, $v^i_1 > v^j_1 + \alpha p(M_n)$ and $v^{\ell+1}_1$ is larger than qualities of all schools in markets $1, \ldots, \ell$. This condition guarantees that it is better to be alone at school v^i, $i > \ell$ than to go to a more worse school v^j, $j < i$, with the best possible peer. Under the above conditions, in any stable matching we must have that students from market n, $\{M_n, M_n\}$ stay home and attend v^n_1. Further, students from market $n-1$ also stay home and attend v_1^{n-1} and so on until market $\ell+1$. We conclude that the matching for the markets $\ell+1,\ldots,n$ is independent from the one for the markets $1,\ldots,\ell$. Therefore, for the rest of the proof we can and will consider only the markets $1,\ldots,\ell$.

The remaining proof consists of three separate cases depending on the structure of the cycle.

Case I (directed cycle): First, suppose that the cycle is directed, so that (without loss of generality) it has the following structure: $1 \to 2 \to \cdots \to \ell \to 1$. Since our cycle is the smallest, there are no other links between the first ℓ vertices of G.

Take $M > \ell+1$ to be specified later, and consider the following sets of types: $F_1 = \{M, 0\}, F_2 = \{M-1, M-3\}, \ldots, F_\ell = \{M-\ell+1, M-\ell-1\}$[13]. Therefore, in any market $1 < i < \ell$ there is one type greater than any one in F_{i+1} and one type which lies between types in F_{i+1}.

We specify switching costs $c_{i,i+1}$, $i = 1, 2, \ldots, \ell$,[14] so that

$$\alpha(p(M) - p(M-\ell-1)) < c_{i,i+1} < \alpha(p(M-\ell-1) - p(0)). \tag{1}$$

Note that the growth condition on $p(\cdot)$ given in Definition 13 guarantees that if M is large enough, then such costs $c_{i,i+1}$ always exist. Next choose arbitrary $v > \max_{i=1,\ldots,\ell} c_{i,i+1}$ and set $v_1 = \cdots = v_\ell = v$.

This finishes the specification. We now prove that no stable matching exists. To begin with, notice that the first inequality in (1) guarantees that for each student it is better to be matched in the home market with any peer except 0, rather than to go to another market: in the home market i the utility for the student from this market is at least $\alpha p(M-\ell-1) + v$ and in the market $i+1$ the utility of the student from i is at most $\alpha p(M) + v - c_{i,i+1}$, which is smaller.

Second, we show that it is impossible for both students from market i to be assigned to market $i+1$. Indeed, if this happens, then the higher type born in market $i+1$ would prefer to stay in his home market, and therefore he will push the lower type born in market i out of the market $i+1$.

[13]We do not explicitly write additional zero types in each F_i, as they will not play a role in the argument. In fact, throughout the proof we ignore their existence.

[14]For ease of notation we avoid writing $i \, mod \, \ell$ and assume that $i = \ell+1$ stands for the market 1 in the remaining proof.

Third, we are going to show that in any stable matching all students, except, perhaps, the student 0 from the market 1, must be matched to some school (rather than choose an outside option with zero utility). Choose a market i. If no student from market $i-1$ is assigned to i, then there is nothing to prove (both students from $i+1$ can stay at home, which is better than taking outside option). The only remaining case is that there is exactly one student from market $i-1$, who is assigned to i. In this case, each of the students from market i has two options which are both better than the outside option (by our choice of v): either to take the second seat in the home-school, or to move to the school in the market $i+1$. If $i \neq 1$, then our choice of types implies that any student from i will always be accepted to the school in the market $i+1$, if he/she prefers to go there. Hence, in this case both students from market i should not choose the outside option. The case $i = 1$ needs a separate consideration because of type 0 there — but we do not claim anything for the student 0.

Forth, let us show that the autarky allocation with each student staying in his own market is not stable. Indeed, suppose we have autarky allocation. In that case M will deviate to the market 2 to get a better peer: $c_{12} < \alpha(p(M-3) - p(0))$. Similarly, even if 0 remains unassigned, M will deviate to the market 2.

At this point there is only one remaining possibility: for some $m = 1, 2, \dots, \ell$, one student from the market m moves to market $m+1$, one student from market $m+1$ moves to market $m+2, \dots$, one student from market ℓ moves to market 1. All other students stay in their own markets.

Suppose $m > 1$. Then the student who moved from market m to market $m+1$ should deviate: there is one empty seat in the market m. The second seat at m is occupied by the other student from m, who has non-zero type. Thus, the former student can take the remaining empty seat in the school in his home market, and we have shown above that matching to a schools in one's own market with a non-zero peer is preferable for each student.

Thus, we must have $m = 1$. In that case either type 0 goes to v_1^2 along with one of the market 2's students or 0 stays at home and M moves to market 2. If 0 moves, then the student from market 2 who is assigned to v_1^3 will deviate and not pay costs, since $c_{23} > \alpha(p(M-2) - p(M-3))$. If 0 stays at home, then he is joined by one of the students from market ℓ. Yet then M will deviate to the market 1, since $c_{12} > \alpha(p(M-1) - p(M-\ell-1))$. Thus, there is no stable matching for the constructed configuration.

Case II (undirected cycle): Now suppose that the above cycle is not a directed cycle. There are two possible subcases. Either there is only one vertex with both edges going away from it or at least two of them. Such type of vertex is shown in Figure 5. Note that the case with zero such vertices corresponds to a directed cycle.

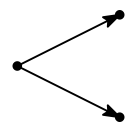

FIGURE 5. Vertex with both edges going away from it.

(1) Suppose that there is only one vertex with both edges going away from it. Then the cycle is shown in Figure 6[15]. That is, there are two directed paths from A to B: one via A_i's and the other via B_j's. Note that a non-directed cycle must have at least three vertices. Those necessary three vertices are $\{A, B, B_1\}$.

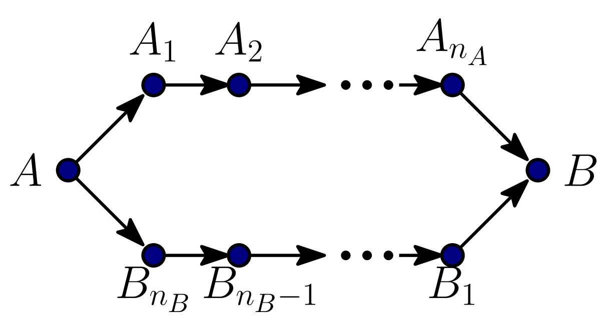

FIGURE 6. Case II.1.

Now let us define the market structure. Choose M large enough and consider the following sets of types[16]:

$$F_A := \{\theta_A, \theta_A\} = \{M - n_A - 1, M - n_A - 1\},$$

[15]For convenience we have changed the names of the markets $1, \ldots, \ell$.

[16]Again, throughout the proof we ignore the existence of additional zeros.

$$F_{A_i} := \{\theta_{A_i}, \theta_{A_i}\} = \{M - (n_A + 1 - i), M - (n_A + 1 - i)\},$$

$$F_B = \{0, 0\},\ F_{B_1} := \{\theta_{B_1}, 0\} = \{M, 0\},$$

$$F_{B_i} := \{\theta_{B_i}, \theta_{B_i}\} = \{M - n_A - 2 - (n_B - i), M - n_A - 2 - (n_B - i)\},\ i \neq 1.$$

That is, the highest type lives in B_1. The next highest types are in A_{n_A}, and types decline with the decrease in A's subscript until we reach the last one, A_1. Among the remaining markets, the highest types are in A, followed by B_{n_B}. Types decline with the decrease in B's subscript until we reach B_2.

Now let us define values of schools:

$$v_1^A = 0,\ v_1^{B_1} = \ldots = v_1^{B_{n_B}} \equiv v,$$

$$v_1^{A_1} = v + \alpha p(M) + 1,\ v_1^{A_2} = v_1^{A_1} + \alpha p(M) + 1 = v + 2\alpha p(M) + 2, \ldots,$$

$$v_1^{A_i} = v_1^{A_{i-1}} + \alpha p(M) + 1 = v + i\alpha p(M) + i, \ldots,$$

$$v_1^B = v_1^{A_{n_A}} + \alpha p(M) + 1 = v + (n_A + 1)\alpha p(M) + n_A + 1.$$

Thus, the highest values are in the markets $A_1, \ldots, A_{n_A}, B$, and they represent an increasing sequence.

Assume all switching costs to be smaller than v, so that it is always better to be assigned to a foreign school than stay unassigned. Moreover, choose costs such that

$$v_1^{A_{n_A}} + \alpha p(M) < v_1^B - c_{A_{n_A}B} + \alpha p(0) \Leftrightarrow c_{A_{n_A}B} < 1 + \alpha p(0). \tag{2}$$

$$v_1^{A_i} + \alpha p(M) < v_1^{A_{i+1}} - c_{A_i A_{i+1}} + \alpha p(0) \Leftrightarrow c_{A_i A_{i+1}} < 1 + \alpha p(0),\ i < n_A. \tag{3}$$

$$\begin{cases} v_1^{B_1} + \alpha p(0) < v_1^B + \alpha p(\theta_{A_{n_A}}) - c_{B_1 B}, \\ v_1^{B_1} + \alpha p(\theta_{B_2}) < v_1^B + \alpha p(\theta_{A_{n_A}}) - c_{B_1 B} \end{cases}$$
$$\Leftrightarrow \begin{cases} c_{B_1 B} < (n_A + 1)\alpha p(M) + n_A + 1 + \alpha(p(\theta_{A_{n_A}}) - p(0)), \\ c_{B_1 B} > (n_A + 1)\alpha p(M) + n_A + 1 + \alpha(p(\theta_{A_{n_A}}) - p(\theta_{B_2})). \end{cases} \tag{4}$$

Moreover, choose $c_{AA_1} = c_{AB_{n_B}} < v$. Then the student θ_A prefers A_1 over B_{n_B} (and both over A), because $v_1^{A_1} + \alpha p(0) = v + \alpha p(M) + 1 + \alpha p(0) > v + \alpha p(M) = v_1^{B_{n_B}} + \alpha p(M)$. Additionally choose $c_{B_iB_{i-1}}$, $i > 1$ such that

(5)
$$\begin{aligned}&\alpha(p(\theta_{B_i}) - p(0)) < c_{B_iB_{i-1}} < v \\ &\Leftrightarrow \alpha(p(M - n_A - 2 - (n_B - i)) - p(0)) < c_{B_iB_{i-1}} < v.\end{aligned}$$ [17]

Eq. (2) guarantees that at least one student from A_{n_A} switches to B. Whether the second student switches depends on the behavior of the student with the highest possible type from the market B_1, M. Eq. (4) guarantees that θ_{B_1} prefers to switch to B and get the peer $\theta_{A_{n_A}}$ instead of staying with zero type at home. However, θ_{B_1} will not switch to B if θ_{B_2} joins him at B_1. Eq. (5) guarantees that all students from markets $B_i, i > 1$ stay at home even with the zero peer.

Suppose first that we have a stable matching where the student $\theta_{B_1} = M$ from B_1 goes to B. Then only one student can switch from A_{n_A}. The second student stays at A_{n_A}. By Eq. (3), we know that one student switches from A_{n_A-1} to A_{n_A}, then one student switches from A_{n_A-2} to A_{n_A-1} and so on, until A_1. Moreover, as students in A prefer A_1 over B_{n_B}, one student from A will switch to A_1. The second one is forced to go to B_{n_B} (it is better than staying at A). Thus, one of the students from B_{n_B} is pushed out. That student moves to B_{n_B-1}, as it is better to be assigned than unassigned. Similarly, the student θ_{B_2} is pushed to B_1. Thus, by Eq. (4), type θ_{B_1} deviates and moves back to B_1.

Now suppose that we have a stable matching where student $\theta_{B_1} = M$ from B_1 does not go to B and stays at home. Thus, by Eq. (2) both students from A_{n_A} switch to B. Similarly both students from A_{n_A-1} switch to A_{n_A} and so on. Finally, both students from A switch to A_1. By Eq. (4), type θ_{B_1} will deviate to B, if his peer is zero. Therefore, he must be with θ_{B_2}. However, as no one from market A goes to B_{n_B} both students from B_{n_B} are not pushed out of their home market. Thus, by Eq. (5), they stay at B_{n_B}. Similarly, by Eq. (5) all students

[17]Existence of such $c_{B_iB_{i-1}}$ can be guaranteed by choosing v high enough.

θ_{B_i}, $i > 1$ stay at home. Thus, θ_{B_2} cannot be at B_1 with θ_{B_1}, and we get a contradiction. Therefore, no stable matching exists for the above economy.

(2) Suppose that there are at least two vertices with both edges going away from them. Then the cycle is shown in Figure 7[18]. That is, nodes with names A_{2k-1} have both edges pointing away from them, nodes with names A_{2k} have both edges pointing towards them, nodes with names B_i^{2k-1} have edges both in the direction of A_{2k} (first towards, then away), and nodes with names B_i^{2k} have edges both in the direction of A_{2k} (first away, then towards).

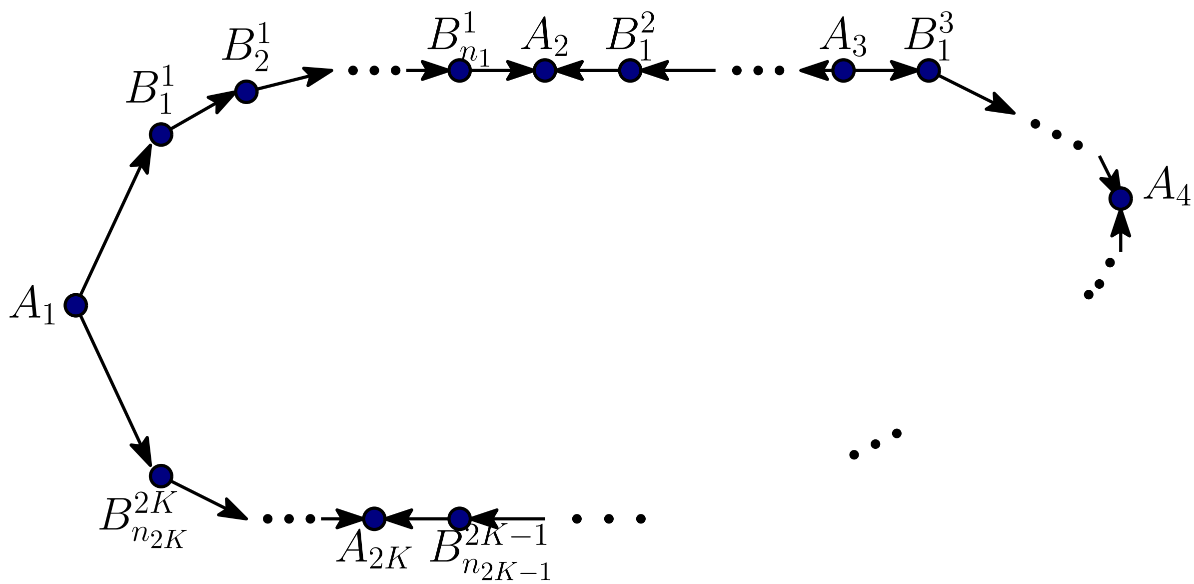

FIGURE 7. Case II.2.

Now let us define the market structure. Choose M large enough, so that[19] $2p(M-K) > p(M)$ and consider the sets of types[20]:

$$F_{A_1} := \{\theta_{A_1}, 0\} = \{M, 0\},\ F_{A_{2l}} = \{0, 0\} \quad \forall l,$$

$$F_{A_{2l+1}} := \{\theta_{A_{2l+1}}, \theta_{A_{2l+1}}\} = \{M-l, M-l\},\ l > 0$$

[18]For convenience we have changed the names of the markets $1, \dots, \ell$.

[19]This can be done as $p(\cdot)$ grows slower than exponentially.

[20]Again, throughout the proof we ignore the existence of additional zeros.

$$F_{B_i^{2l+1}} := \{\theta_{B_i^{2l+1}}, \theta_{B_i^{2l+1}}\} = \left\{\theta_{A_{2l+1}} - \frac{i}{1+n_{2l+1}}, \theta_{A_{2l+1}} - \frac{i}{1+n_{2l+1}}\right\}$$
$$= \left\{M - l - \frac{i}{1+n_{2l+1}}, M - l - \frac{i}{1+n_{2l+1}}\right\},$$

$$F_{B_i^{2l}} := \{\theta_{B_i^{2l}}, \theta_{B_i^{2l}}\} = \left\{\theta_{A_{2l+1}} + \frac{1+n_{2l}-i}{(1+n_{2l})(1+n_{2l-1})}, \theta_{A_{2l+1}} - \frac{1+n_{2l}-i}{(1+n_{2l})(1+n_{2l-1})}\right\}$$
$$= \left\{M - l + \frac{1+n_{2l}-i}{(1+n_{2l})(1+n_{2l-1})}, M - l + \frac{1+n_{2l}-i}{(1+n_{2l})(1+n_{2l-1})}\right\}, \ i \neq K,$$

$$F_{B_i^{2K}} := \{\theta_{B_i^{2K}}, \theta_{B_i^{2K}}\} = \left\{\theta_{A_{2K-1}} - 1 - \frac{i}{1+n_{2K}}, \theta_{A_{2K-1}} - 1 - \frac{i}{1+n_{2K}}\right\}$$
$$= \left\{M - K - \frac{i}{1+n_{2K}}, M - K - \frac{i}{1+n_{2K}}\right\}.$$

That is, the highest type lives in A_1. The larger is the subscript of A_i among odd ones, the smaller are types in that market. Moreover, $\theta_{B^{2l+1}_{n_{2l+1}}} > \theta_{B_1^{2l+2}}$, and types in the intermediate markets, B_i^k, are decreasing in i and they lie on the intervals $(\theta_{A_k} - 1, \theta_{A_k})$ for odd k and $(\theta_{A_{k+1}}, \theta_{A_{k+1}} + 1)$ for even $k \neq 2K$.

Now let us define school values:

$$v_1^{A_{2l+1}} = 0, \ v_1^{B_i^{2l+1}} = v_1^{A_{2l}} \equiv v \quad \forall l, i,$$

$$v_1^{B_i^{2l}} = (n_{2l} - i + 1)\alpha p(M - l + 1) + n_{2l} - i + 1 \quad \forall l, i,$$

where v is large enough so that $v > (n_{2l} + 1)\alpha p(M - l + 1) + n_{2l} + 1$ for all l.

Assume all switching costs to be smaller than v, so that it is always better to be assigned to a foreign school than stay unassigned. Set $c_{A_1 B^{2K}_{n_{2K}}} = 0$. Moreover, choose costs such that (we write $B^{2l+1}_{n_{2l+1}+1}$ for A_{2l+2} and B_0^{2l} for A_{2l})

$$\begin{aligned} &v_1^{B_{i-1}^{2l}} + \alpha p(0) - c_{B_i^{2l} B_{i-1}^{2l}} > v_1^{B_i^{2l}} + \alpha p\left(M - l - \frac{1+n_{2l}-i}{(1+n_{2l})(1+n_{2l-1})}\right) \\ &\Rightarrow c_{B_i^{2l} B_{i-1}^{2l}} < \alpha\left(p(M-l+1) - p\left(M - l - \frac{1+n_{2l}-i}{(1+n_{2l})(1+n_{2l-1})}\right) + p(0)\right) + 1, \end{aligned} \tag{6}$$

$$\begin{aligned} &v_1^{B^{2l}_{n_{2l}}} + \alpha p(0) - c_{A_{2l+1} B^{2l}_{n_{2l}}} > v_1^{A_{2l+1}} + \alpha p(M - l) = \alpha p(M - l) \\ &\Rightarrow c_{A_{2l+1} B^{2l}_{n_{2l}}} < \alpha\left(p(M - l + 1) - p(M - l) + p(0)\right) + 1, \end{aligned} \tag{7}$$

(8)
$$v_1^{B_1} + \alpha p\left(M - \frac{1}{1+n_1}\right) - c_{A_1B_1^1} > v_1^{B^{2K}_{n_{2K}}} + \alpha p(0) - c_{A_1B^{2K}_{n_{2K}}}$$
$$\Rightarrow \begin{cases} c_{A_1B_1^1} < v - \alpha p(M-K-1) - 1 + \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p(0), & n_{2K} \neq 0, \\ c_{A_1B_1^1} < \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p(0), & n_{2K} = 0 \end{cases},$$

(9)
$$v_1^{B_1} + \alpha p\left(M - \frac{1}{1+n_1}\right) - c_{A_1B_1^1} < v_1^{B^{2K}_{n_{2K}}} + \alpha p\left(M - K - \frac{n_{2K}}{1+n_{2K}}\right) - c_{A_1B^{2K}_{n_{2K}}}$$
$$\Rightarrow \begin{cases} c_{A_1B_1^1} > v - \alpha p(M-K-1) - 1 + \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p\left(M - K - \frac{n_{2K}}{1+n_{2K}}\right), & n_{2K} \neq 0, \\ c_{A_1B_1^1} > \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p(M-K+1), & n_{2K} = 0, \end{cases},$$

(10)
$$v_1^{B_i^{2l+1}} + \alpha p(0) > v_1^{B_{i+1}^{2l+1}} + \alpha p\left(M - l - \frac{i}{1+n_{2l+1}}\right) - c_{B_i^{2l+1}B_{i+1}^{2l+1}}$$
$$\Leftrightarrow c_{B_i^{2l+1}B_{i+1}^{2l+1}} > \alpha\left(p\left(M - l - \frac{i}{1+n_{2l+1}}\right) - p(0)\right).$$

Note that Eq. (9) and Eq. (10) do not contradict the initial requirement that $c < v$, as because $p(\cdot)$ grows slower than exponentially, we can find M such that $2p(M-K) > p(M)$ and $v > v - \alpha p(M-K-1) - 1 + \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p\left(M - K - \frac{n_{2K}}{1+n_{2K}}\right)$. Moreover, by choice of v, as $v_1^{B_i^{2l+1}} = v$, we get $v_1^{B_i^{2l+1}} > (n_{2l}+1)\alpha p(M-l+1) + n_{2l} + 1 > \alpha\left(p\left(M - l - \frac{i}{1+n_{2l+1}}\right) - p(0)\right)$.

Because $v_1^{A_1} = 0$, θ_{A_1} does not stay at home in any stable matching. Suppose first that in a stable matching θ_{A_1} goes to B_1^1. Then he/she pushes out $\theta_{B_1^1}$. Note that the second $\theta_{B_1^1}$ stays at home, as home guarantees a better peer and no switching costs, while school value is the same at home and abroad. Thus, only one $\theta_{B_1^1}$ goes to B_2^1. Similarly, only one $\theta_{B_2^1}$ goes to B_3^1 and so on. Finally, only one $\theta_{B_{n_1}^1}$ goes to A_2. By Eq. (6) both $\theta_{B_1^2}$ want to switch to A_2, but only one can, as $\theta_{B_{n_1}^1} > \theta_{B_1^2}$. Thus, one $\theta_{B_1^2}$ switches and another stays at home. Similarly, one $\theta_{B_2^2}$ switches to B_1^2 and another stays at home and so on. Thus, one θ_{A_3} switches to $B_{n_2}^2$ and another to B_1^3 and so on. Finally, we are left with the fact that one $\theta_{B^{2K}_{n_{2K}}}$ switches to $B^{2K}_{n_{2K}-1}$ and another stays at home. Then by Eq. (9), θ_{A_1} deviates and switches to $B^{2K}_{n_{2K}}$.

Now suppose that θ_{A_1} goes to $B^{2K}_{n_{2K}}$. Then by Eq. (10), no one switches from B^1_1. Similarly no one switches from B^1_2 and so on until $B^1_{n_1}$. Thus, by Eq. (6), both students switch from B^2_1 to A_2. By the same logic, both students switch from B^2_2 to B^2_1 and so on. Finally, by Eq. (7), both students switch from A_3 to $B^2_{n_2}$. Applying the same arguments to the next markets, we get that students from $B^{2K-1}_{n_{2K-1}}$ stay at home, while students from B^{2K}_1 switch to A_{2K} and, at the end, students from $B^{2K}_{n_{2K}}$ switch to $B^{2K}_{n_{2K}-1}$. Thus, θ_{A_1} is left alone at $B^{2K}_{n_{2K}}$. However, by Eq. (8), in that case θ_{A_1} deviates and switches to B^1_1. Thus, there is no stable matching for the constructed configuration. □

Proof of Proposition 4. In the proof we assume that no two students share the same type θ, and no student is indifferent between two schools. That is, first, there do not exist students θ from market i and θ' from market j such that $\theta = \theta'$, and, second, there do not exist a triple of markets i, j, f and two schools (ℓ, i) and (k, j) such that $v^i_\ell - c_{fi} = v^j_k - c_{fj}$.

Denote the equilibrium from the student-proposing (thus, student-optimal) Gale-Shapley algorithm as eq_1, and suppose there exists another equilibrium eq_2. Let us prove that $eq_2 = eq_1$. When students propose to their most preferred schools, schools start to accept the highest types. The student with $\theta = \max\{F_1 \cup \ldots \cup F_n\}$ is admitted for sure (this is the most preferred type for schools). Then the second highest type is admitted for sure and so on until some school reaches its capacity. Suppose school v^i_1 reaches its capacity first (then those, who were proposing to that school and were not accepted need to propose to another school). Denote by $\underline{\theta}^i_1$ the smallest type admitted to v^i_1.

We claim that all types $\theta \geq \underline{\theta}^i_1$ have the same allocations under both equilibria. Suppose by contradiction, that there exist a type θ_0 from a market i_0, such that he/she has different allocations in eq_1 and eq_2. Under eq_1 he is admitted to the first best option, $v^{i(\theta_0)}_1$, thus, under eq_2 he is admitted to a different school. If he can switch to $v^{i(\theta_0)}_1$, he would do so. Thus, he must be below the cut-off for $v^{i(\theta_0)}_1$, and the school must be filled up to capacity. Thus, $\theta_0 < \underline{\eta}^{i(\theta_0)}_1$, where $\underline{\eta}$ denote cut-offs under eq_2. If he is not accepted, it means that there are other students, which have types above $\underline{\eta}^{i(\theta_0)}_1$ and are assigned to $v^{i(\theta_0)}_1$, while they were assigned to a different school under eq_1. Thus, we get a student θ_1 from a market i_1 who prefers to switch to his first best from eq_1, $v^{i(\theta_1)}_1$, but is assigned to $v^{i(\theta_0)}_1$. Thus, his first best school is filled up to capacity and has threshold $\underline{\eta}^{i(\theta_1)}_1 > \theta_1$. Because the number of schools is finite, we can repeat the process until we get $k, \ell, k > \ell$ such that $\theta^k > \theta^\ell$, and the first best option for the student θ^k from the market $i(k)$ is $v_1^{i(\theta_k)}$, but he is assigned to a different school $\left(v_1^{i(\theta_{k-1})}\right)$, while θ^ℓ wants to go to $v_1^{i(\theta_\ell)}$ but is admitted to $v_1^{i(\theta_{\ell-1})} = v_1^{i(\theta_k)}$ under eq_2. Thus, θ^k can profitably deviate, and we get a contradiction. Therefore, all types $\theta \geq \underline{\theta}_1^i$ have the same allocations under both equilibria. Applying the same procedure to the next cut-off in eq_1, we get that the allocations coincide. Because number of schools, and, thus, cut-offs is finite, we get that $eq_1 = eq_2$. Thus, the equilibrium is unique. □

Proof of Proposition 5. Let us proceed in the following way: first, reorder in any way schools in each market, then apply the algorithm from Theorem 1 for the new school ordering. We will get a different matching, because now not schools with the highest v's are getting the best peers in each market, but some other schools. For example, if we reorder schools from last to first, across each market we will see the best students in the schools with the smallest values.

Let us show that no one will deviate from the above assignment. Because v's and c'c now do not matter, no one is going to deviate across markets (all schools, which can accept a given student have lower peers than one's current assignment). Thus, the only rreasonable deviation can be to a different market. However, the construction in Theorem 1 guarantees that it is also non-profitable. □

Example 3. *("Envy-free matching without many zero types") One of the possible ways to construct an envy-free matching with a bunch of fully occupied schools and without many zeros assumption is to run the algorithm from Theorem 1 and add a Pre-Step. Every time we go back to Step* 0 *we first do the following:*

Pre-Step: For each market m, check that for the current best school in that market, v_1^m, there are enough students to take all its seats (with many zero types that was always the case). That is, we need to check that $q_1^m \leq |F_m| + \sum_{m' \colon \{m' \to m\} \in G} |F_{m'}|$, where $|F|$ is the number of elements in F. For all markets m where the condition fails, we delete all schools in m (they are left empty) and proceed to Step 0 *with the new economy.*[21]

[21] We do not change the assignments to schools in m, which were matched with students in the previous rounds of the algorithm. By construction, they are occupied up to capacity by non-zero types.

As an outcome, we get a matching where each school either has no empty seats or has no students at all. By construction, it is envy-free (the same argument as was used in the proof of Theorem 1 to establish stability ensures envy-freeness here). Moreover, all the properties of the initial algorithm remain valid and we get a matching, which is assortative inside markets, but not across markets. There are matchings where more students are assigned to schools. We can modify the above procedure and combine it with serial dictatorship: we run serial dictatorship as described above for the set of unmatched students and empty schools. This decreases a number of students assigned to an outside option. Yet, potentially it can still be possible to come up with a different matching, where even less students are matched to the outside option.

Proof of Theorem 6. Consider the algorithm from Theorem 1. We add a Pre-Step, which happens before Step 0 every time we run the algorithm.

Pre-Step: For each market m choose the highest type $\bar{\theta}(m)$ among those who can go to a school at m, i.e. $\bar{\theta}(m) = \max\left\{F_m \bigcup_{m' \colon \{m' \to m\} \in G} F_{m'}\right\}$. For any school (ℓ, m), consider an assignment A_ℓ^m, which fills the school (ℓ, m) up to capacity with the best students from $F_m \bigcup_{m' \colon \{m' \to m\} \in G} F_{m'}$ (i.e. from the market m and markets m' such that $\{m' \to m\} \in G$). Student $\bar{\theta}(m)$ gets a seat in all k_m assignments (k_m is the number of schools in the market m). Choose $\ell^*(m)$, such that $\bar{\theta}(m)$ gets the highest utility in the $A^m_{\ell^*(m)}$ among all k_m assignments. Then use the school $(\ell^*(m), m)$ instead of the school $(1, m)$ in the algorithm.

That is, in the next steps of the algorithm we try to fill up to capacity one of the schools $v^1_{\ell^*(1)}, \ldots, v^n_{\ell^*(n)}$ instead of $v^1_1, \ldots, v^n_1$. After some school is filled, we delete that school and its students and return to Pre-Step.

We need to show that $\ell^*(m)$ is well-defined. If there are two students with the type $\bar{\theta}$ and potentially different home markets who can get a seat in market m, then they agree on the most preferred school, as they have the same peers: for both of them the set of peers in a school is $\Theta \cup \{\bar{\theta}\}$, i.e. they are each other's peers and the rest of the class are peers for both of them. Thus, both students choose $\arg\max_k \{v_\ell^m + \alpha p(\bar{\theta}, \text{other top students}) - (\text{switching cost to } m)\} = \arg\max_k \{v_\ell^m + \alpha p(\bar{\theta}, \text{other top students})\}$, which does not depend on the origin of the student. Thus, $\ell^*(m)$ is well-defined.

Define the outcome of the algorithm as μ. Let us explain why μ ia a group-stable matching. Suppose by contradiction that (C, s), where $s = (\ell, m)$, blocks μ. Then $C \cap \Theta(\ell, m, C; \mu)$, where $\Theta(\ell, m, C; \mu) = \{q_\ell^m \text{ highest types from } C \cup \mu^{-1}(\ell, m)\}$, also blocks μ. Without loss of generality let us assume that $C = C \cap \{q_\ell^m \text{ highest types from } C \cup \mu^{-1}(\ell, j)\}$, that is, all students from C get seats at s after deviation to s.

Consider a student $\theta \in C$, such that in the above algorithm θ was the first to get a seat among members of C. If there are multiple such students, then choose anyone with the highest type among those in C who got their seats earliest. Suppose μ assigns θ to some school (ℓ^*, m^*), where $\ell = k_m + 1$ represents the outside option. Let us first show that if $m^* = m$, then θ cannot benefit from such deviation.

If $m^* = m$, then θ can only deviate to schools in m, which were filled after (ℓ^*, m), because by construction schools which were filled earlier are occupied by higher types. Thus, θ cannot push anyone out of the schools, which were filled earlier in the process. Thus, school (ℓ^*, m) was chosen by the highest type among $F_m \bigcup_{\{m' \to m\} \in G} F_{m'}$ at the Pre-Step corresponding to the round of the algorithm when θ got a seat. Thus, the highest type was getting weakly larger utility at (ℓ^*, m) than at any other currently not yet deleted school in m and tentatively filled with highest possible types, including (ℓ, m). So, if θ was that highest type, $\bar{\theta}(m)$, then he/she cannot benefit from deviating to (ℓ, m): all students from C get a seat at the same time as θ or later, thus, they were available when θ was making comparisons. Therefore, peer set for θ after deviation is the same as it was at the moment of comparison or even worse. So θ's utility cannot strictly increase from deviation to (ℓ, m) (it is not larger than θ's utility in the Pre-Step from attending (ℓ, m), which is weakly less than θ's utility from (ℓ^*, m), i.e. from μ). If θ was not the highest type, who chose (ℓ^*, m) among other schools in m, then that highest type is not a member of C (otherwise θ wont be the highest type from C among those who were assigned seats the earliest in the algorithm, as $\bar{\theta}(m)$ gets a seat at the same time as θ). Then, if θ deviates to (ℓ, m), $\bar{\theta}(m)$ is not there, and the peer set for θ after deviation is the same as it was at the moment of comparison for the highest type $\bar{\theta}(m)$ or even worse. Thus, θ gets weakly lower utility at (ℓ, m) than $\bar{\theta}(m)$ was getting at the Pre-Step. While at (ℓ^*, m), θ gets weakly larger utility than $\bar{\theta}(m)$. Thus, $u_\theta\left(\mu(\theta), \mu^{-1}\mu(\theta) \setminus \{\theta\}\right) \geq u_{\bar{\theta}(m)}\left(\mu(\theta), \mu^{-1}\mu(\theta) \setminus \{\bar{\theta}(m)\}\right) \geq u_{\bar{\theta}(m)}(A_\ell^m) \geq u_\theta\left(v_\ell^m, \text{peers after deviation}\right)$. Therefore, θ does not benefit from deviation to (ℓ, m).

Second, we need to show that if $m^* \neq m$, then θ cannot benefit from such deviation. As before, if (ℓ, m) was already filled, when θ was assigned a school, then (ℓ, m) is filled by higher types. Otherwise, as θ can switch to market m, it would also be assigned a seat at (ℓ, m). By construction, in each round of the algorithm we fill a school with the highest types, who are allowed to go to the school's market. Thus, if θ can push someone out of (ℓ, m), this school cannot be filled prior to (ℓ^*, m^*), where θ got a seat.

When we were fixing assignment to (ℓ^*, m^*), we must have asked the current highest type from θ's home market, $\bar{\theta}(\theta)$, whether he/she prefers assignment to (ℓ^*, m^*) or to $(\ell^*(m), m)$. The highest type $\bar{\theta}(\theta)$ must have chosen (ℓ^*, m^*). Suppose that $\ell^*(m) = l$. Thus, $u_{\bar{\theta}(\theta)}(v_{\ell^*}^{m^*}, \text{best peers}) \geq u_{\bar{\theta}(\theta)}(v_{\ell}^{m}, \text{best peers})$ and $\theta \neq \bar{\theta}(\theta)$, as $\bar{\theta}(\theta)$ cannot be a member of C (he/she does not benefit from deviation to s). If C deviates to s, then θ's set of peers is at most the same as $\bar{\theta}(\theta)$ had while choosing (ℓ^*, m^*), because we know that $\bar{\theta}(\theta)$ wont be at s. Thus,

$$u_\theta\left(\mu(\theta), \mu^{-1}\mu(\theta) \setminus \{\theta\}\right) \geq u_{\bar{\theta}(\theta)}\left(\mu(\theta), \mu^{-1}\mu(\theta) \setminus \{\bar{\theta}(\theta)\}\right) \geq u_{\bar{\theta}(\theta)}\left(v_{\ell}^{m}, \text{best peer} \setminus \{\bar{\theta}(\theta)\}\right)$$
$$\geq u_\theta\left(v_{\ell}^{m}, \text{best peer} \setminus \{\bar{\theta}(\theta)\}\right) \geq u_\theta(\text{coalition } C \text{ deviates to } s).$$

Thus, θ cannot profitably deviate to $(\ell^*(m), m)$.

We are left with the case $m^* \neq m$, $\ell^*(m) \neq l$. Because $\bar{\theta}(\theta) \geq \theta$ and because $\bar{\theta}(\theta)$ preferred $v_{\ell^*}^{m^*}$ to $v_{\ell^*(m)}^{m}$ (otherwise we would not fix the assignment to (ℓ^*, m^*)),

$$u_\theta\left(v_{\ell^*}^{m^*}, \mu^{-1}\mu(\theta) \setminus \{\theta\}\right) \geq u_{\bar{\theta}(\theta)}\left(v_{\ell^*}^{m^*}, \mu^{-1}\mu(\theta) \setminus \{\bar{\theta}(\theta)\}\right) \geq u_{\bar{\theta}(\theta)}\left(v_{\ell^*(m)}^{m}, \text{best peers} \setminus \{\bar{\theta}(\theta)\}\right).$$

Moreover, because $\bar{\theta}(m)$ chose $\ell^*(m)$, $u_{\bar{\theta}(m)}(A_{\ell^*(m)}^{m}) \geq u_{\bar{\theta}(m)}(A_{\ell}^{m})$ and $\bar{\theta}(m)$ cannot be a member of C. Because $\bar{\theta}(\theta) \leq \bar{\theta}(m)$ (highest type inside one market is smaller than highest type across this market plus some other markets),

$$u_{\bar{\theta}(\theta)}\left(v_{\ell^*(m)}^{m}, \text{best peers} \setminus \{\bar{\theta}(\theta)\}\right) \geq u_{\bar{\theta}(m)}\left(v_{\ell^*(m)}^{m}, \text{best peers} \setminus \{\bar{\theta}(m)\}\right) + c(\bar{\theta}(m)) - c(\bar{\theta}(\theta)),$$

where $c(\theta)$ is the switching cost the student θ has to pay to go to the market m. Because θ and $\bar{\theta}$ are from the same home market, $c(\bar{\theta}(\theta)) = c(\theta)$. Thus,

$$u_\theta\left(\mu(\theta), \mu^{-1}\mu(\theta) \setminus \{\theta\}\right) \geq u_{\bar{\theta}(m)}\left(v_{\ell^*(m)}^{m}, \text{best peers} \setminus \{\bar{\theta}(m)\}\right) + c(\bar{\theta}(m)) - c(\bar{\theta}(\theta))$$
$$\geq u_{\bar{\theta}(m)}\left(v_{\ell}^{m}, \text{best peers} \setminus \{\bar{\theta}(m)\}\right) + c(\bar{\theta}(m)) - c(\theta) = u_{\bar{\theta}(m)}(A_{\ell}^{m}) + c(\bar{\theta}(m)) - c(\theta).$$

So if θ deviates to s, then his best possible peers are those, who were assigned to A_ℓ^m without $\bar\theta(m)$ (we already argued that $\bar\theta(m)$ is not going to deviate). Therefore, $u_\theta(\text{coalition } C \text{ deviates to } s) \leq u_{\bar\theta(m)}(A_\ell^m) + c(\bar\theta(m)) - c(\theta)$ and $u_\theta(\text{coalition } C \text{ deviates to } s) \leq u_\theta\left(\mu(\theta), \mu^{-1}\mu(\theta) \setminus \{\theta\}\right)$. Thus, θ cannot profitably deviate to (ℓ, m). □

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