STABILITY IN MATCHING MARKETS WITH PEER EFFECTS

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Abstract. The paper investigates conditions which guarantee the existence of a stable outcome in a school matching in the presence of peer effects. We consider an economy where agents are characterized by their type (e.g. SAT score), and schools are characterized by their value (e.g. teaching quality) and capacity. Moreover, we divide agents and schools into groups, so that going to a school outside of one's group may be associated with additional costs or even prohibited. A student receives utility from a school per se (its value minus costs of attending) and from one's peers, students who also go to that school. We find that sufficient condition for a stable matching to exist is that a directed graph, which governs the possibility to go from one group to another, should not have (directed or undirected) cycles. We also construct a polynomial time algorithm, which produces a stable matching. Furthermore, we show that if the graph has a cycle, then there exist other economy parameters (types, costs and so on), so that no stable matching exists. In addition, in cases where a stable matching exists we investigate whether it is unique.

1. Introduction

Peer effects are a common phenomenon in everyday life. Parents often try to place their kids in schools where they believe that their children will have good classmates. That is, parents care not only about quality of teachers and curriculum, but also about whom is going to study with their children. Similarly, many students want to go to Ivy League universities because of the connections that they will likely make at such places.

The presence of peer effects in schooling was noticed more than fifty years ago (see e.g. Coleman et al. (1966, Section 2.4)). Sacerdote (2011) provides an overview of the current state of empirical research on peer effects and points to its importance. A number

Date: December 4, 2018.

The author would like to thank Larry Samuelson for valuable comments, suggestions, and encouragement throughout the duration of the project. The author is grateful to Vadim Gorin for support and fruitful discussions.

of recent papers show the significance of peer effects in schooling (yet, the magnitude of the importance of peer effects varies across papers). Examples are Ding and Lehrer (2007) (peer effects in China), Sacerdote (2001) (peer effects at Dartmouth), Winston and Zimmerman (2004) (peer effects in US colleges), Zabel (2008) (peer effects in New York), Zimmerman (2003) (peer effects at Williams College). For more examples and details see also Epple and Romano (2011), which provides a survey of recent papers on various types of peer effects in education.

When we go to theory, the relationship between schools or colleges and students is usually modelled as a two-sided matching problem. In matching models without peer effects and externalities, substitutability is a sufficient (and in some sense necessary) condition for the existence of a (group) stable matching (see Gale and Shapley (1962), Hatfield and Milgrom (2005) and Hatfield and Kojima (2008)). Unfortunately, matching models with peer effects are known to often lack existence of equilibria. This motivates us to study theoretical models of matching in the presence of peer effects.

Several authors investigated the existence of stable matchings in the presence of peer effects in the recent years. Two prevalent approaches are either doing an algorithmic search for the equilibrium (e.g. Echenique and Yenmez (2007)) or imposing some structural restrictions on the model, which lead to an automatic existence of an equilibrium (e.g. Bodine-Baron et al. (2011), Pycia (2012)). These and other related articles will be discussed in more detail below. Despite the progress, there is still no simple criterion for the existence of stable matchings, which would apply to wide enough class of models. From the theoretical viewpoint, one would like to find out what basic features of preferences can guarantee existence (or non-existence) of the equilibrium in the presence of peer effects.

Our paper makes a step in this direction. We do not look at the most general choices of preferences, as it is hardly possible to say anything at all, if no structure is present in the model (see, however, section 7 for some generalizations). Instead, we propose and analyze a model which captures two main features: the utility of a student depends on the peers matched to the same school through a (quite general) peer effect function, and schools have intrinsic qualities and are arranged in districts, with moving costs arising when a student wants to change his/her district. Many authors agree that both peer effects and subdivision into districts have to be present in any realistic school matching model (see, e.g., Calsamiglia et al. (2015) and references therein), and, therefore, it naturally leads to studying the effect of these two basic features on the existence of the equilibrium.

We start by modifying the college admission model, which was studied in the seminal paper of Gale and Shapley (1962), and add preferences over schoolmates. Consequently, students now care both about their assigned school and their peers. This is modelled as a linear combination of school-related utility and utility from a given set of peers. We focus on a pairwise stable matchings; in the context of schools this means that no single student can profitably deviate to another school which would accept him. We believe pairwise stability to be a natural assumption in case of schools, where a parent cannot easily coordinate with other parents and place their children in the same school.

The division of schools into districts is based on the following real life phenomena. Sometimes an agent may be prohibited from applying to particular schools. For example, religious schools generally accept only those students, who practice the same religion. Moreover, to go to a Jewish school, one often needs to present a proof of one's Jewish roots. Similarly sometimes schools accept only those, who live in specified areas. Thus, students, who live outside of those areas cannot be admitted. A large set of schools in Moscow function in that way. They can be viewed as district-specific as they admit only those who live close enough. Finally, segregation corresponds to the structure, where some agents are restricted from some set of schools. Instead of schools we can think about specific majors. Then it may be too late (and, thus, impossible) to switch from studying, e.g., ballet to studying quantum physics. Those patterns can be encoded into a graph. Possibility/impossibility to move from one group to the other corresponds to the presence/absence of an edge between the groups, which correspond to graph vertices.

Let us describe the **general setting** in more detail. In our model preferences of schools coincide: they prefer students who have higher type (e.g. higher test scores). We allow more flexibility on the students' side. We divide the set of students into groups, similarly, we assign each school to one of those groups. All agents from the same group have the same valuations of schools.

Such division can be viewed as different markets. That is, being in one group means being from the same market such as country, race, religion, specialization, etc. A school attached to a group is located in the same market as students from that group. For example, they all are in the same city. Then the difference between how an agent from a market values a school from the same market and how an agent from a different market values that school is expressed as an additional "market switching" cost c. Such cost is location and origin specific, so that we still have the same preferences across markets. We can view this cost as the expenses associated with buying an apartment near that school or with commuting costs or with costs of switching from one field of primary study to the other (e.g. switching from mathematically inclined school to the one which focuses more on humanities).

To sum up, we get a set of separate markets, where students only differ by their ability or type, but do not differ in their preferences of schools. Moreover, going to a school in a foreign market is associated with additional costs for an agent born in a different home market. Obviously, in some cases such cost may be prohibitively high, so that there is no way an agent from a market X can attend a school in a market Y (e.g. religious schools for someone outside of that religion or legal segregation of schools in the US in the 20th century). We can summarize that prohibition by drawing an oriented graph, where vertices represents our groups/markets, and an edge from one vertex to another means that switch from the former to the latter is not prohibited. Such prohibitively high costs will be crucial for our results. What would matter for our constructions and conclusions is the oriented switching graph, not the exact values of intermediate, not prohibitive switching costs.

Our main result provides conditions for the existence of a stable matching in the above model. We find that the sufficient condition is that there are no cycles (neither directed, nor undirected) in the directed graph of possible market switches: when the switching graph is an oriented forest, we present an algorithm, which produces a stable matching. Further, we show the necessity of "no cycles" condition: if there is a cycle (possibly undirected), then there are parameters for which there is no stable matching. We refer to section 2 for an illustrative example of the importance of acyclicity condition in the simplest case when the graph has two vertices. Our main results are given in Theorems 1 and 2. We also discuss when a stable matching is unique/non-unique (see Propositions 4 and 5).

To our knowledge, the most novel aspect of our condition lies in the non-directness. The classical results on the existence of a stable matching prohibit only directed cycles. E.g. in a roommate problem (Gale and Shapley (1962)) lack of directed cycles in agents' preferences guarantees stability. Non-directed cycles were not playing a major role before, however for our setting they are of the same importance as directed cycles.

Related papers, which investigate the existence of stable matching with peer effects, are Pycia (2012), Echenique and Yenmez (2007), and Bodine-Baron et al. (2011). The first paper provides a condition (pairwise alignment of preferences), which guarantees the existence of a core stable (and, thus, also pairwise stable) matching. This condition and ours are non-nested. In our setting, pairwise alignment means that if we assign two students, say a and b, to some school and some set of peers and then consider a different assignment, where again a and b are at the same school, they must agree on whether the former or the latter allocation is better. However, such condition is not satisfied in our framework: a and b may disagree even if they were born in the same market, because they have different set of peers (a is in the set of peers of b, but not in the set of peers of itself), and this distinction may be of different importance depending on how large the school is. When school is small having one better peer means more than when school is large. So that even if the quality per se of a smaller school is worse, a may still prefer it: e.g. if b is very good peer, a may want to choose a small school, where there will be almost no one except itself and b. But if the second school is much better than the first, and is filled with agents similar to a, b may choose a second, larger school. Thus, preferences are not aligned, and our model still leads to an open question. The second paper, Echenique and Yenmez (2007), presents an algorithm, which produces a set of allocations containing all stable matchings in case they exist. However, it requires searching for a fixed point of a certain operator over the set of all possible matchings, and implementing such an algorithm may be very time consuming (in fact, in some cases it leads to just checking all possible allocations). In contrast, we provide specific conditions for a stable matching to exist, so that we do not need to check different possible allocations. It would be interesting to study whether structural restrictions similar to the ones considered in our text can lead to the significant decrease of the running time of the algorithm of Echenique and Yenmez (2007). The third paper, Bodine-Baron et al. (2011), suggests a model for matching in the presence of peer effects arising from the underlying social network. In their model a properly defined equilibrium always exists. However, the key assumption guaranteeing this is a certain symmetry condition for the social network graph — no analogue of this condition is present in our setting.

Other related literature

One of the important aspects of our setting is subdivision into districts. The idea that sometimes agents have to choose from subsets of possible matching partners (i.e. they belong to a smaller market with fixed subset of alternatives) or there is a separate matching technology inside each market has been present in the market design literature. Though, the most common question differs from ours: the literature mostly concerns whether merging two markets into one can be beneficial for agents.

For example, Ortega (2018) considers the matching between men and women when agents are partitioned into disjoint groups. The question is then whether there is a way to improve the inside-groups matching by allowing to match agents outside of their groups. He shows that a stable matching across the whole population can not hurt more than half of the society compared to the inside-groups outcome. Doğan and Yenmez (2017) investigate the market, where only one side, schools, is divided into groups. Students are not restricted to a specific group and can apply to any school. Each group of schools runs a separate matching algorithm, so that at the end some students get offers from multiple groups, while some do not get any offers at all. The authors propose a unified enrollment system which turns out to be better for students. It works jointly with all schools and leads to at most one offer to each student. Manjunath and Turhan (2016) propose an alternative approach to improve upon the outcome of the independent across school groups admission process. They suggest to do an iterative rematching after the independent processes are done.

Similarly, Nikzad et al. (2016) look at the possibility of merging two markets into one in the context of kidney exchange. One of the markets represents the US, where kidney exchange is well developed, but there is lack of donors, and the other represents a developing country with almost no suitable medical facilities, but with willing donors. The authors show that merging those markets into one will increase the welfare in US.

On the peer effect side, the coalition formation literature such as Bogomolnaia and Jackson (2002), Banerjee et al. (2001), and Kaneko and Wooders (1986) is relevant. If one views schools as additional agents and ask players to form coalitions, additionally assuming that coalition with more than one or zero schools will lead to a utility of a negative infinity, we get precisely the problem of finding a stable coalition. However, our model does not satisfy conditions from the above papers to ensure existence of a stable matching.

The rest of the paper is organised as follows. Section 2 presents a motivating example, which illustrates the main ideas of the paper. Section 3 builds up the model and defines our solution concept, pairwise stability. Section 4 provides the sufficient condition (no cycles) for the existence of a pairwise stable matching, while section 5 shows that that condition can be viewed as necessary: for any graph G with a cycle there exist set of other parameters (types, school values, etc.), so that in the corresponding economy no stable matching exists. Section 6 talks about uniquenes/non-uniqueness of a stable matching, when it exists. Section 7 discusses the role, which our assumptions play in obtaining the results, and possible generalizations. Finally, section 8 concludes. All proofs are in the Appendix.

2. Illustrative example

Consider the example, which illustrates the model and the associated existence problem.

Suppose we have two schools, $\mathcal{A}$ and $\mathcal{B}$, and each school has two seats. There are four students characterized by their type (e.g. test score) $\theta = 0, 7, 8, 10$. Schools prefer students with higher types, and utility of an agent θ sharing a school s with another student θ' is

$$u_\theta(s, \theta') = v_\theta(s) + \theta'.$$

If θ is alone at school s, then $u_\theta(s, \emptyset) = v_\theta(s)$. Utility of the school per se, $v_\theta(s)$ is

$v_\theta(s)$:

θ \ s	$\mathcal{A}$	$\mathcal{B}$
0, 10	10	5.5
7, 8	6	9.5

That is, 0 and 10 prefer school $\mathcal{A}$, while 7 and 8 prefer school $\mathcal{B}$. The interpretation is that 0 and 10 live in the district of school $\mathcal{A}$, and therefore, going to $\mathcal{B}$ leads to additional switching costs, thus, decreasing utility. Similarly, 7 and 8 live in the district of school $\mathcal{B}$.

The example is in some sense similar to a classical roommate problem with two rooms and four agents, one of whom no one likes (Gale and Shapley (1962)). Here we have a zero type, whom no one wants as a peer, as it means zero peer effects. Although 10 is the best possible peer, it is still not worth to switch to a less desirable school to join 10, if the most favorite one has a "normal" (i.e. 7 or 8) peer.

There are no pairwise stable matchings. The argument, summarized in Figure 1, is:

- If $(8, 10) \to \mathcal{A}$, then $7 \to \mathcal{B}$, so that 8 deviates to $\mathcal{B}$:

$$u_8(\mathcal{A}, 10) = 10 + 6 = 16 < 9.5 + 7 = 16.5 = u_8(\mathcal{B}, 7);$$

- Similarly, if $(7, 10) \to \mathcal{A}$, then 7 deviates;
- If $(8, 10) \to \mathcal{B}$, then $7 \to \mathcal{A}$, so that 10 deviates to $\mathcal{A}$:

$$u_{10}(\mathcal{B}, 8) = 5.5 + 8 = 13.5 < 10 + 7 = 17 = u_{10}(\mathcal{A}, 7);$$

- Similarly, if $(7, 10) \to \mathcal{B}$, then 10 deviates;
- If $(7, 8) \to \mathcal{B}$, then $10 \to \mathcal{A}$, so that 10 deviates to $\mathcal{B}$:

$$u_{10}(\mathcal{A}, 0) = 10 + 0 = 10 < 5.5 + 8 = 13.5 = u_{10}(\mathcal{B}, 8);$$

- Similarly, if $(7, 8) \to \mathcal{A}$, then 10 deviates;

Thus, there are no stable matchings in the above economy. Note that our example corresponds to a full graph (each agent can go to any school), so it has a cycle.

Now suppose that it is prohibitively costly for student 10 to attend school $\mathcal{B}$, perhaps because student 10 is the wrong religion or gender, or faces prohibitive relocation costs.

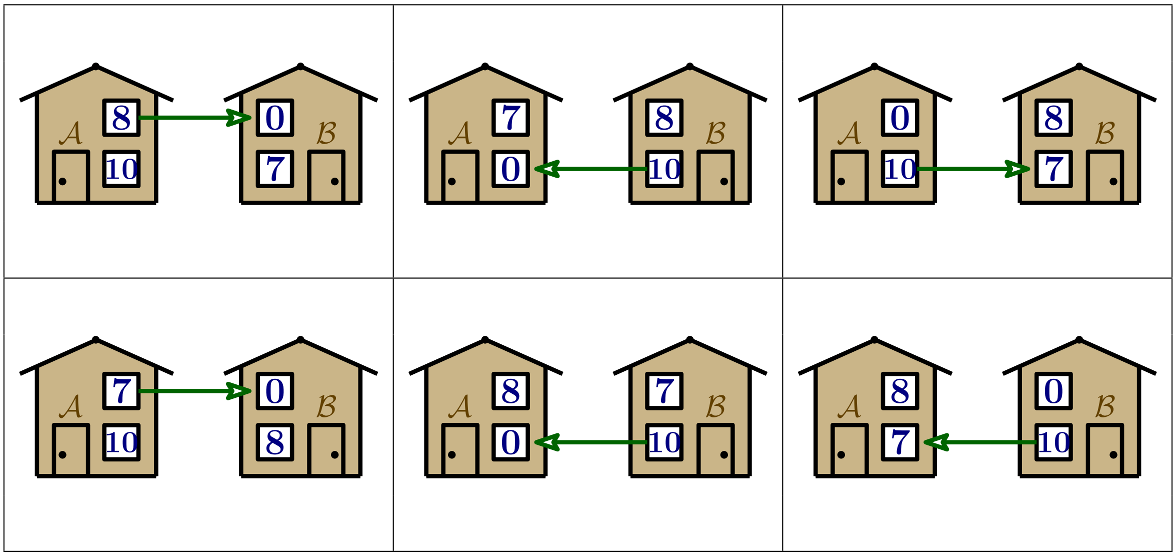

FIGURE 1. No stable matchings in the illustrative example. Arrows represent profitable deviations.

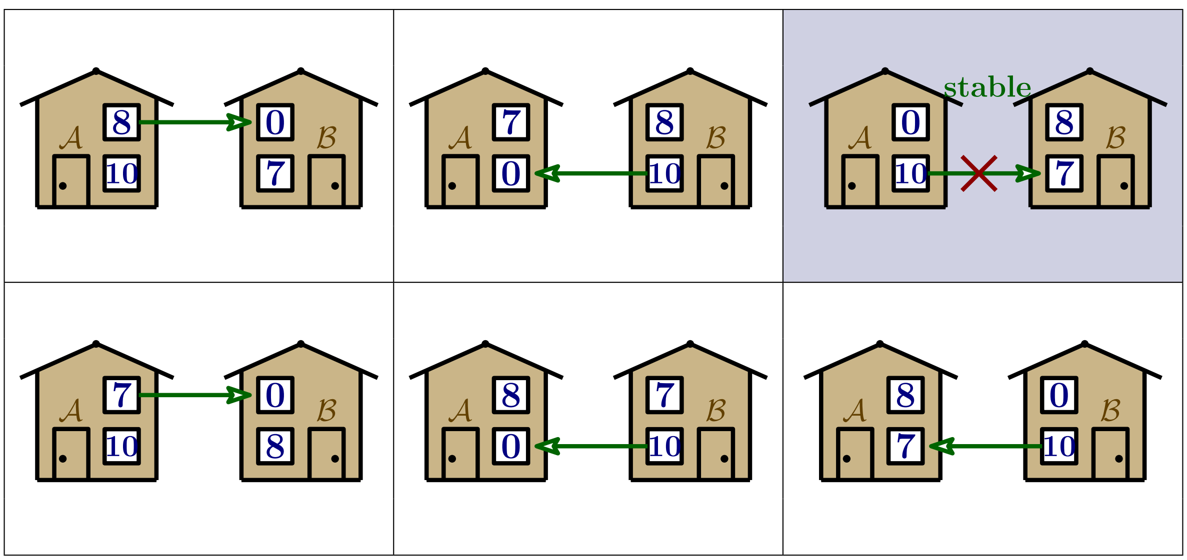

FIGURE 2. Stable matching in the illustrative example.

Then the existence of a stable matching is restored. We assign $0, 10$ to $\mathcal{A}$ and $7, 8$ to $\mathcal{B}$. 10 is not allowed to deviate, and we get a stable matching, as summarized in Figure 2. On the graph side, the prohibitive cost means that our graph now has a directed edge from the market of school $\mathcal{B}$ to the market of school $\mathcal{A}$, but not in the opposite direction. Thus, there is no cycle.

3. Basic model

3.1. **Setting.** Let us consider a world with n markets. Each market i has k_i different schools. In any market i, any school ℓ has capacity[1] $q^i_\ell \geq 2$ and is associated with utility v^i_ℓ. It cannot exceed its capacity for students and would like to take as many students below capacity as possible. Moreover, schools prefer students with higher ability. Without loss of generality we number schools in each market by their attractiveness, i.e. we assume $v^i_1 \geq v^i_2 \geq \ldots \geq v^i_{k_i}$ for all $i = 1, \ldots, n$. Without loss of generality we may also assume that the best school is located in market 1, that is we assume $v^1_1 \geq v^i_1$ for all $i = 2, \ldots, n$. To shorten the notation, we also sometimes use v^i_ℓ to denote the school itself: this is the school ℓ in the market i. When it is important to stress that we are referring to school's name, not its attractiveness level, we will use the notation (ℓ, i)

Additionally, each market i is populated with m_i students of different abilities[2]. Changing one's initial market is costly for the students. The possibility to switch between markets is governed by a directed **switching graph** $G = (V, E)$, where markets are vertices. That is, the graph G consists of n vertices ($|V| = n$). Edges represent the possibility to switch from one market to another. If $\{i \to j\} \in G$, then it is allowed to switch from market i to market j, although the switch may be associated with some costs. If $\{i \to j\} \notin G$, then it means that market j is infeasible to agents born in market i. That is, either it is too costly for them to attend (even the best allocation in j would not offset switching costs) or it is just prohibited by some underlying laws. The interpretation of the switching graph G is discussed in the Introduction and at the end of Section 7.

Each student is characterized by type θ and home market i. Let F_i denote all types of students (with repetitions) in market i. Formally each F_i is a multiset (see Definition 1). We assume that all types are non-negative and each F_i is finite and has a large enough number of zero types. For example, having $z_i = \sum\limits_{l=1}^{k_i} q^i_l + \sum\limits_{j:\,\{i\to j\}\in G} \sum\limits_{l=1}^{k_j} q^j_l$ zeros in each market i is enough, though, in practice we will need even smaller number of zero types.

[1]We may allow for capacity of one with minor modification to the Algorithm in Theorem 1.

[2]Most of our results also hold for continuous distributions, so that instead of m_i students we will have mass m_i of students. For example, the algorithm in Theorem 1 is still valid: if higher types do not deviate, then lower types also stay. Discreteness is only used in the construction in Theorem 2.

The idea is that zero types help us avoid partially filled schools. They fill seats, which would be otherwise empty. We discuss the zero types assumption and its role in more detail in Section 7.

The difference across students in different markets comes from the fact that if a student from market i wants to change one's initial market and apply to a school in a different (but feasible, i.e. such that $\{i \to j\} \in G$) market j, the student has to bear additional cost $c_{ij} \geq 0$, where $c_{ii} = 0$. This can be viewed as a travelling costs of going to a foreign market (e.g. additional time it takes every morning to go to a further located school). Alternatively we can view those costs as psychological losses from being far from one's family and/or being surrounded by people from a different background. This is in a sense a mismatch penalty. There can be a number of other interpretations of costs.

Each student has an outside option with 0 utility (i.e. not attend a school). If student does attend a school, then one's utility from attending a school is composed of school's own effect v_ℓ^j and a peer effect. The peer effect is described by a peer-effect function $p(\cdot)$, defined on all real multisets. Our basic model assumes that peer-effect function does not depend on one's own type, however, we discuss some possible generalizations in this direction in Section 7.

Definition 1. *A **real multiset** is a finite collection of reals, in which we allow the same number to be repeated arbitrary many times. The order of the elements of the multiset is irrelevant. Let $m[\mathbb{R}]$ denote the family of all real multisets.*

Definition 2. *A **peer-effect function** is a non-negative function $p : m[\mathbb{R}] \to \mathbb{R}_+$, such that $p(\emptyset) = 0$ and $p(\cdot)$ is increasing. By increasing we mean that for any multiset Θ if $\theta' \geq \sup \Theta$ ($\theta' \leq \inf \Theta$), then $p(\Theta \cup \{\theta'\}) \geq p(\Theta)$ ($p(\Theta \cup \{\theta'\}) \leq p(\Theta)$), and if $\theta' > \theta$, then $p(\Theta \cup \{\theta'\}) \geq p(\Theta \cup \{\theta\})$.*

For a student ϑ of type θ in market i by its peers we always mean the multiset of types of all other students attending the same school as ϑ.

Finally, utility for a student ϑ of type θ in market i from attending a school v_ℓ^j is

$$u_\vartheta(v_\ell^j,\ \text{peers of } \vartheta) = v_\ell^j + \alpha \cdot p(\text{peers of } \vartheta) - c_{ij},$$

where the coefficient $\alpha \geq 0$ measures the importance of peer effects. When it leads to no confusion, we use $u_{\theta,i}$ instead of u_ϑ.

The average quality of one's peers is the natural example of a peer-effect function. It satisfies our assumptions, and we will use it quite frequently. Moreover, in some sense it corresponds to an approach in empirical research where a student's outcome Y (e.g. test scores or alcohol use) linearly depends on the average of background characteristic (types in our case) of one's peers (see, for example, review article Sacerdote (2011)).

Denote by $s(\vartheta)$ the school, where student ϑ goes, and by Θ the multiset of types all peers of that student. That is, $\Theta = \{\text{type of } \vartheta' \mid \vartheta' \neq \vartheta, \quad s(\vartheta) = s(\vartheta')\}$. Then the average quality of ϑ's peers is:

$$p(\Theta) = \frac{\sum\limits_{\theta' \in \Theta} \theta'}{\sum\limits_{\theta' \in \Theta} 1},$$

where term θ' appears in the sum as many times as it appears in the multiset Θ.

Two other common examples of a peer effect function would be the best and the worst types: $p(\Theta) = \sup \Theta$ and $p(\Theta) = \inf \Theta$. Similarly, we can do an average of, say, two best or two worst students.

3.2. **Stable matching.** We are interested in pairwise stable matchings, so that no student-school pair can profitably deviate and match together.

Definition 3. *A **matching** is a mapping μ from set of all students into set of all schools, such that for each school ℓ in each market i,*

$$|\mu^{-1}(\ell, i)| \leq q_\ell^i,$$

where $|M|$ stands for the number of elements in the set M. That is, schools cannot accept above their capacities.

Definition 4. *A matching μ is **individually rational** if for any agent ϑ,*

$$u_\vartheta(\mu(\vartheta), \mu^{-1}(\mu(\vartheta)) \setminus \{\vartheta\}) \geq 0.$$

That is, no one prefers being unmatched to one's assignment under μ.

Definition 5. *A matching μ is **feasible** if for each agent ϑ in each market i, $\mu(\vartheta) = (\ell, j)$ implies that $\{i, j\} \in G$.*

Namely, feasibility implies that each agent is matched to a school in a market into which one is allowed to switch.

Define a set of peers, which one gets after a deviation to a school v_ℓ^j under a matching μ as

$$\Theta(\ell, j; \mu) = \begin{cases} \mu^{-1}(\ell, j), & \text{if } |\mu^{-1}(\ell, j)| < q_\ell^j; \\ \mu^{-1}(\ell, j) \setminus \{\min(\mu^{-1}(\ell, j))\}, & \text{if } |\mu^{-1}(\ell, j)| = q_\ell^j. \end{cases}$$

Thus, if the school v_ℓ^j is full, and an agent ϑ deviates to that school, ϑ pushes away a student with the lowest type.

Definition 6. *A feasible matching μ is **stable** if it is individually rational and for any agent ϑ of type θ in market i:*

$$\text{if } u_\vartheta(\mu(\vartheta), \mu^{-1}(\mu(\vartheta)) \setminus \{\vartheta\}) < v_\ell^j + \alpha \cdot p(\Theta(\ell, j; \mu)) - c_{ij}, \text{ then}$$
$$|\mu^{-1}(\ell, j)| = q_\ell^j \text{ and } \theta \leq \min(\mu^{-1}(\ell, j)).$$

The above means that for each student, all more preferred schools are filled up to capacity by students with higher types.

In the following two sections we are going to first propose a sufficient condition on the graph of available market switches, G, which guarantees the existence of a stable matching. Then we will show that our condition is in a sense necessary, that is if G has cycles, then it is possible to find types, costs, and school values and capacities, such that no stable matching would exist.

4. Sufficiency

In this section we present a sufficient condition, which guarantees the existence of a stable matching. Under our condition, there exists an algorithm, which produces a stable matching. Some properties of that algorithm and associated stable outcome are investigated below. We also compare our sufficiency condition with the pairwise alignment condition of Pycia (2012).

4.1. **Construction of a stable matching.** The example, presented in the Introduction, illustrates that non-existence of stable matchings may come from the possibility of agents cyclically switching their locations: an agent X, born in market i_1, moves to market i_2 and pushes an agent Y away from a school in his home market i_2, so that Y needs to switch market. The agent Y switches the market from i_2 to i_1, so that market i_1 becomes better, and X prefers to stay at home and not pay extra travelling costs. When X moves home, Y can go back, as his previous seat is now empty. Y returns to i_2 and we are back to the start of the cycle. This is summarised in the Figure 3.

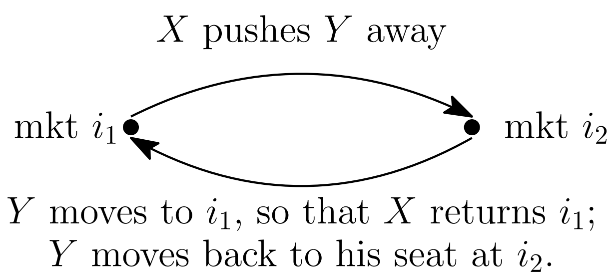

FIGURE 3. Cycle of length 2.

Similar pattern may arise with multiple market switch. E.g., if someone moves from market i_1 to market i_2 and pushed other agent away, that other one moves from i_2 to i_3, and so on until an agent is pushed from i_l and moves to i_1. That makes i_1 attractive again, so that the first agent returns, leaving an empty seat at i_2. Then the second agent returns and so on.

Moreover, even non-directed cycles like $i_1 \to i_2 \to i_3$, $i_1 \to i_3$ may cause a problem: an agent from i_1 may go to i_3, which is the most desired place for an agent from i_2, so that the agent from i_2 now cannot go to i_3 (it is full). However, when the agent from i_2 is at i_2, the agent from i_1 may decide to stay with him at i_2, thus, leaving the seat at i_3 vacant. Therefore, the agent from i_2 will take it and leave the agent from i_1 alone at i_2. So the agent from i_1 will switch to i_3 and push the other agent back to i_2, and we get a cyclical pattern, which prohibits the existence of a stable matching.

The following theorem proves that as long as no cycles exist in the switching graph G, a stable matching exists.

Definition 7. *A **tree** is an undirected graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree.*

Definition 8. *An **oriented tree** is a directed graph whose underlying undirected graph is a tree.*

Definition 9. *An **oriented forest** is a disjoint union of finite number of oriented trees.*

Theorem 1. *Suppose that the switching graph G is an oriented forest. Then a stable matching exists. Such matching can be found by a finite iterative algorithm.*

The proof in the Appendix presents an algorithm that constructs a stable match for any oriented forest.

The algorithm consists of several iterations: in each iteration we fill some school up to its capacity and further remove this school and all its students from the considerations. Let us say that this is a school ℓ in a market i. In each iteration, it is crucial that we manage to find an assignment of students to the school (ℓ, i) satisfying the following three properties. First, (ℓ, i) is the best school in its market i. Second, the students assigned to (ℓ, i) are of the highest types among all the students from $F_i \bigcup_{\{j \to i\} \in G} F_j$. Third, if the best student $\bar{\theta}_j$ from F_j got assigned to (ℓ, i), then for each other assignment to a school from $\{j\} \bigcup \{j' \mid \{j' \to j\} \in G\}$, which satisfies the first two properties, $\bar{\theta}_j$ should also be a part of this assignment, but his utility should be (weakly) less. That is, $\bar{\theta}_j$ should get a seat in the best schools in all markets, where he is allowed to switch, and should weakly prefer assignment to (ℓ, i).

It is non-trivial that such a school exists at all, however, the acyclicity of G guarantees the existence. The stability of the matching constructed through such iterations would follow from the following observation: in the third property we only need to look at the highest student type in each market. Lower student types will agree to follow the highest one. See Appendix for the details of the proof.

Let us discuss some *properties of the algorithm and of the stable outcome* it produces.

Importantly, the choice of a starting market does not change the outcome of the algorithm. That is, it does not matter the best school in which market we fill the first. We show that in Appendix in Lemma 1.

Next, the matching, which we get in the algorithm from Theorem 1, has an assortative pattern: inside each market, agents are allocated to schools in an assortative manner. That is, the better is school in market i, the higher types have students assigned to that school. Formally,

$$\forall i, \ell, \ell' \text{ s.t. } \ell < \ell' \text{ if } \theta \text{ is matched to } v_\ell^i \text{ and } \theta' \text{ is matched to } v_{\ell'}^i, \text{ then } \theta \geq \theta'.$$

Such construction serves as an instrument to make deviations inside a given market unprofitable.

Further, for a given oriented tree, at each round of the algorithm we take at most n steps (the worst is if we go from the root to a leaf covering all other $n-1$ markets). Then at each round we fill one school (including an outside option). Thus, in total we will need at most $n(k_1 + 1 + k_2 + 1 + \ldots + k_n + 1) = n\left(n + \sum_{i=1}^{n} k_i\right)$ units of time.

Let us remark that the presence of peer effects may preclude the existence of student-optimal or school-optimal matchings.[3] Consider, for example, the following model with 2 markets. Suppose that $G = \{2 \to 1\}$ and $c_{21} = 1$. There are two schools in market 1: $v_1^1 = 8$, $v_2^1 = 7$, and no schools in the market 2. Students are $F_1 = \{10, 4\}$ and $F_2 = \{9, 5\}$. Then there are two stable matchings: $(9, 10) \to v_1^1, (5, 4) \to v_2^1$ and $(9, 10) \to v_2^1, (5, 4) \to v_1^1$. The above algorithm picks the first one. Students disagree on which is better: 10 and 9 prefer the first matching, while 5 and 4 prefer the other one. The idea is that now having a high enough peer can force a student to stay at a lower ranked school. Thus, although 10 and 9 prefer v_1^1 per se, if they can not coordinate on going to it, it is better to them to stay together at v_2^1.

Similarly, the presence of peer effects can make comparative statics ambiguous. In our case the only obvious change is a marginal increase in a quality of some school or a marginal increase in someone's type. The small increase is important, as it guarantees that an allocation per se does not change. In that case those, who share an allocation at the improved school or with the made better peer gain, while others are not affected. However, when the increase is large enough to change the matching, we get some peers who can get this better match and those, who are now pushed away and, thus, lose.

[3]A student-optimal (school-optimal) matching is a stable matching which all students (schools) weakly prefer to any other stable matching.

Likewise, for example, if we increase capacity for some school, someone else will be able to get into it and, thus, will benefit. Yet others are now suffering from additional low peer.

4.2. **Comparison with Pycia (2012).** The questions in Pycia (2012) are closely related to ours. The author investigates necessary and sufficient conditions for the existence of a group stable matching in the matching model with peer effects. His crucial condition is pairwise alignment of preferences. This means that if we fix two agents and consider any two assignments, under both of which those agents share the same coalition, then they must agree on which assignment is better. This requirement and ours are non-nested.

In one direction it is clear: if we consider a full graph on two vertices with very small switching costs c_{12}, c_{21}, and very small α, then for values v_ℓ^i such that differences $v_\ell^i - c_{ij}$ are large enough pairwise alignment is satisfied (costs and peer effects are negligible, so only schools qualities v_ℓ^i matter, and agents agree on them). However, such graph has a cycle.

For another direction consider the following example. When there is only one market, $n = 1$, our algorithm leads to the assortative matching[4]. In that case agents agree on which school is the best, and, thus, if we match the best students with the best school, there will be no reason to unilaterally deviate from such assignment. However, the case of only one market still can violate pairwise alignment condition for group stability of Pycia (2012).

The violation comes from the fact that different agents in the same school can get different peer effects, as they have different set of peers (agent a is in the set of peers of agent b, but not in the set of peers of oneself). For example, let $\alpha = 1$ and let "average peer" be the peer-effect function. Suppose we have two schools with values 10 and 9.5. The first school has capacity 3, while the second has capacity 2. We consider

[4]We remark that assortative matching does not have to be a unique equilibrium. For example, the allocation where highest types are assigned to the second best school up to capacity and the remaining students and schools are matched assortatively is stable for α large enough. High types do not deviate from the second best school because this leads to a significant decrease in the peer effects. For more details see Section 6.

$a = 5$, $b = 0$, and fill the remaining seat at the first school with additional zero. Then $u_a(10, \{0, 0\}) = 10 > 9.5 = u_a(9.5, \{0\})$, while $u_b(10, \{5, 0\}) = 12.5 < 14.5 = u_b(9.5, \{5\})$. Thus, a and b disagree on which assignment is the best, and their preferences are not pairwise aligned.

Group stability is a more demanding condition than the pairwise stability. Yet, our algorithm applied to schools $v_1^1, \dots, v_{k_1}^1$ filled with the top students instead of $v_1^1, \dots, v_1^n$ will produce a group stable outcome for the case $n = 1$.

5. Necessity

In this section we show that if a directed graph G of available market switches has cycles (not necessary directed), then there exists a set of parameters, for which there is no stable matching. Theorem 2, which is proved in the Appendix, summarizes the result.

Definition 10. *We say that an increasing function $f(x)$ of $x \geq 0$* ***grows slower than exponentially*** *if*

$$\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = 1.$$

For instance, the function x^k for any $k > 0$ grows slower than exponentially.

Theorem 2. *Assume that $p(\{x\})$ is strictly increasing but grows slower than exponentially as a function of $x \in \mathbb{R}_+$. If, ignoring edge directions, G has a cycle, then there exist values of $\{v_k^i\}_{i,k}$, $\{c_{ij}\}_{i,j}$, $\{F_i\}_i$ such that the resulting economy has no stable matching.*

Remark 1. *We need to assume that peer-effect function is not constant. Otherwise agents do not care about their peers: they get the same constant utility from any set of peers. So we are left with a model without peer effects, and the classical Gale-Shapley algorithm will produce a pairwise stable matching. The assumption that $p(\{x\})$ is strictly increasing as a function of $x \in \mathbb{R}_+$ helps us to get rid of the above.*

Remark 2. *Slow growth of function $p(\{x\})$ is a technical condition, which guarantees the existence of switching costs satisfying certain inequalities. We stick to slower than exponential growth to simplify the solution. We believe that Theorem 2 will also hold without imposing it.*

The construction in the proof is in the spirit of the Illustrative Example from the Introduction. We put the highest type M and the lowest type, 0, in the same originating market. We choose costs such that the highest type would prefer to stay at some market, say i (either home or foreign market), with non-zero type, but will deviate to a different market, say j, if one has to share a seat with 0 at the market i while j guarantees a non-zero peer. Then if M goes to the market j, it eventually leads to some non-zero type going to the market i, so that M can go to his best choice, i. Similarly, if M goes to the market i, then positive types do not join M there, so that he is left with 0 as a peer, and deviates to the market j.

6. Uniqueness/non-uniqueness of a stable matching

In the previous sections we have seen that when G has no cycles, stable matchings exist. However, we have not explored whether there is only one stable matchings or there are many of them. In this section we will answer the question of uniqueness/non-uniqueness of stable matchings for the two boundary cases: "no peer effects" ($\alpha = 0$) and "only peer effects" (α large enough).

We will show that when there are no peer effects, a stable matching can be found by applying Gale-Shapley algorithm (Gale and Shapley (1962)), and it is generically unique.[5] In contrast, when peer effects dominate, so that only one's classmates matter we get a multiplicity of equilibria.

6.1. No peer effects.

We can calculate the equilibrium by iterative matching of the best schools and the most high-skilled students (that is, we apply student-proposing Gale-Shapley algorithm and break indifferences in an arbitrary way). When $\alpha = 0$ we get a special case of a model of Gale and Shapley (1962), where prohibition to go to a market can be interpreted as having a large negative utility from schools in that market. Thus, we are guaranteed the existence of a stable matching.

[5]The Lebesgue measure zero set of student types, school values, and switching costs such that there are indifferences of the form $v^i_\ell - c_{fi} = v^j_k - c_{fj}$ or there are two agents of the same type may lead to multiple stable matchings.

Proposition 3. *If $\alpha = 0$, then for any graph structure G there exists a stable matching in the above model.*

We can have more than one stable matching in two cases. First, if there are two or more agents of the same types θ (possibly from different originating markets), so that a school does not know whom to accept for the last available seat. Second, if some agent is indifferent between two schools, so that this agent does not have exactly one best option to which to point in the above construction. That is, as long as, first, there do not exist agents θ from a market i and θ' from a market j such that $\theta = \theta'$, and, second, there do not exist a triple of markets i, j, f and two schools (ℓ, i) and (k, j) such that $v_\ell^i - c_{fi} = v_k^j - c_{fj}$, the equilibrium is unique. The Lebesgue measure of the set

$$\left\{ \left\{v_\ell^i\right\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,k_i}}, \left\{c_{ij}\right\}_{i\neq j}, \left\{\theta_\ell^i\right\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,m_i}} : \theta_\ell^i = \theta_k^j \text{ or } v_\ell^i - c_{fi} = v_k^j - c_{fj} \text{ for some } f, (i,\ell) \neq (j,k) \right\}$$
$$\in \mathbb{R}^{\sum_{i=1}^n k_i} \times \mathbb{R}^{n(n-1)} \times \mathbb{R}^{\sum_{i=1}^n m_i}$$

is zero. So generically a stable matching is unique.

The intuition for uniqueness is that if we look at the most high-skilled student among those, who play different strategy compared to the above equilibrium, then this student is going to a worse school. That happens because in the above equilibrium one is guaranteed the best choice among those which are not occupied by higher types, so deviating to the strategy from the above equilibrium will be beneficial (we assume no indifferences). Proposition 4 summarizes uniqueness results and is proved in the Appendix.

Proposition 4. *If $\alpha = 0$, then for any graph structure G and numbers of schools and students per district the stable outcome of the above model is generically unique. That is, the Lebesgue measure of the subset of*

$$\left\{ \left\{v_\ell^i\right\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,k_i}}, \left\{c_{ij}\right\}_{i\neq j}, \left\{\theta_\ell^i\right\}_{\substack{i=1,\dots,n \\ \ell=1,\dots,m_i}} \right\} \in \mathbb{R}^{\sum_{i=1}^n k_i} \times \mathbb{R}^{n(n-1)} \times \mathbb{R}^{\sum_{i=1}^n m_i}$$

for which there are multiple stable matchings is zero.

6.2. **Only peer effects.** The next proposition illustrates that when α becomes large enough, so that peer effects dominate, and v's and c's become unimportant, the situation becomes a coordination problem. High types would like to coordinate and stay together, and they have different possibilities on which to coordinate. Such different possibilities give us multiple equilibria. The idea is that instead of trying to put the best types in schools with the highest values we can try to put them, for example, in the schools with lowest values, and they still will not deviate, as they are getting the highest possible peer effects.

Proposition 5. *If α is large enough, G has no cycles, and at least one market has at least 2 schools, then there are multiple stable matchings.*

7. Role of the assumptions and extensions

In this section we examine what role are various assumptions of the model playing, how important they are, and how generalizable they are. First let us talk about the assumption on F_i. We impose that it has a large number of 0s. We use it to get rid of only partially filled schools (completely empty schools do not cause a problem). The following example illustrates why a partially filled school may cause a problem for the existence of a stable matching even when the switching graph has no cycles.

Example 1. *("empty seats") Suppose that $\alpha = 1$ and there are 2 markets and 1 school per market, capacities are $q_1^1 = 2$, $q_1^2 = 3$, school values are $v_1^1 = 1$, $v_1^2 = 1$, student types are $F_1 = \{1, 10\}$, $F_2 = \{11\}$, the switching graph is $G = \{1 \to 2\}$, and the corresponding switching cost is $c_{12} = 7$. Thus, it is impossible to move from market 2 to market 1, and we do not have cycles. However, there still does not exist a stable matching. Capacities are such that agents can always be admitted to their home school, thus, no one will choose outside option ($1 > 0$). Possible matchings are*

- *If $(1, 10) \to v_1^1$, then 10 deviates to v_1^2:*

$$u_{10}(1) = 1 + 1 = 2 < 1 + 11 - 7 = 5 = u_{10}(2);$$

- *If $1 \to v_1^1$, $(10, 11) \to v_1^2$, then 1 deviates to v_1^2:*

$$u_1(1) = 1 < 1 + 10.5 - 7 = 4.5 = u_1(2);$$

- *If $(1, 10, 11) \to v_1^2$, then 10 deviates to v_1^1:*

$$u_{10}(1) = 1 > 1 + 6 - 7 = 0 = u_{10}(2);$$

Thus, there are no stable matchings in the above economy.

Zero types help to get rid of non-existence, because then if an agent switches to some school, the peer effect from that school can only go up (someone with a lower type is pushed away). In contrast, with empty seats low types can switch and decrease the peer effect. In Example 1 this happened when 1 switched to the second market.

Example 1 illustrates that if there is a partially filled school in a market i and it is possible to switch from market j to i, then the existence of a stable matching may fail. However, we do not need to impose zero types in the markets, to which no one can switch (i.e. $\nexists j$ s.t. $\{j \to i\} \in G$). This is because in the algorithm in Theorem 1, when we compare different allocations and choose the most preferred one for the highest types, we need to know that if one does not want to go to some school ℓ in market j, then one will not want to go to that school later (e.g. we cannot have a situation where 10 prefers to stay at home with 0 more than being abroad with 0 and 11, but after we fix such assignment, 10 wants to join 11 assuming 0 remains at home). If later the school will have empty seats, others may want to join (as 10 joins 11). However, if no one can switch to market j, by monotonicity inside markets of the algorithm, only low types will stay at the partially filled school ℓ, so that higher types do not have incentive to go back. When higher types were choosing whether to stay at home or not, they were looking at even better peer-set at home, and still decided to leave.

The second crucial assumption is that agents inside any market have the same preferences, and agents from different markets i and j still agree on the relative order of schools in any given market. The former guarantees us that lower types do not deviate from an assignment as long as higher types of the same origin also stay. The latter guarantees that highest types from different origins agree on the best school inside any market and, thus, if placed in that school, do not wish to deviate to a different school inside that market.

It is possible to relax the assumption of identical preferences of agents from the same origin. We can assume that utility of an agent θ who was born in a market i and attends a school v_ℓ^j with peers Θ is:

$$v_\ell^j + \alpha \cdot p(\Theta) - c_{ij} - c(\theta, i),$$

where for all $i, j, \ell, \Theta, \theta > \theta'$ if $v_\ell^j + \alpha \cdot p(\Theta \cup \theta') - c_{ij} - c(\theta, i) \geq 0$, then $v_\ell^j + \alpha \cdot p(\Theta \cup \theta) - c_{ij} - c(\theta', i) \geq 0$. This can be satisfied if, for example, $c(\theta, i)$ is an increasing function of θ, so that higher types also have higher costs. Alternatively, for discrete economy we have finite set of equations for $c(\theta, i)$ and can choose any solution to those. Such generalization allows different agents born in the same market to have different preferences. Yet, the relative utility between two different schools still remains the same. The proof of Theorem 1 is still valid in this setting. (We only need to add outside option as one more alternative to compare for each of the highest type, as now for large enough value of $c(\theta, i)$ a high type may have negative utility even from the best school and, thus, prefer to stay unmatched.)

The other way to generalize preferences is to go from linear function of $v, c,$ and $p(\Theta)$ to an arbitrary function $u_{\theta,i}(v_k^j, c_{ij}, \Theta)$. Assuming that u is increasing in the first argument, v, decreasing in the second argument, c, and is a peer-effect function for any fixed v, c (see Definition 2). Then for $\theta_i \leq \bar{\theta}_i$, if $\bar{\theta}_i$ does not deviate from v_k^j to $v_{k'}^{j'}$, θ_i also stays at v_k^j:

$$u\left(v_k^j, \Theta \setminus \theta_i \cup \bar{\theta}_i, c_{ij}\right) \geq u\left(v_k^j, \Theta, c_{ij}\right) \geq u\left(v_{k'}^{j'}, \Theta', c_{ij'}\right).$$

Thus, Theorem 1 is still valid. Similarly, Theorem 2 if stated in terms of function u instead of p, remains valid.

Moreover, we can allow some dependence on one's own type for the peer effect function. That is, working with $\tilde{p}(\theta, \Theta) = p(\Theta) + f(\theta)$, where $f(\cdot)$ is non-negative and weakly decreasing still guarantees the existence of a stable matching. Weak decrease in θ can be interpreted in a sense that lower types need more guidance and, thus, benefit more from better sets of peers, while higher types are more independent and, thus, care less about their peers. For such peer-effect function the main idea of Theorem 1 still holds: if higher type prefers one school over the other, than so do low types ($f(\theta)$ cancels out when we make a comparison). The only difference is that now if higher type prefers to stay unmatched, lower types may still prefer to go to a school, as they have higher value of $f(\cdot)$. Thus, if some high type $\bar{\theta}_i$ prefers an outside option to all currently available schools, we can not say that so do all agents below some $\bar{\theta}$, who are born in the same market i. For some $\theta' < \bar{\theta}_i$ born in market i we may get $v_\ell^j + \alpha(p(\Theta) + f(\theta')) - c_{ij} > 0 > v_\ell^j + \alpha(p(\Theta \setminus \{\bar{\theta}\} \cup \theta') + f(\bar{\theta})) - c_{ij}$. Thus, we will need to leave all agents with types below or equal to θ' born in market i

in the mechanism for further steps. Those agents may eventually be matched to some school, where they get positive utility.

Finally let us analyze the switching graph G. Sometimes, as in the examples with religious or district schools we have it as given. There may be cases, when there are no explicit restrictions on who can apply to a given set of schools. Yet, if for some group of students, i, the utility associated with another group of schools, j, even in the best possible matching (best school plus best peers) is less than switching costs (e.g. exams are too hard so that it is not worth an effort), then we can impose the condition that $\{i \to j\} \notin G$, which will not affect possible matchings. Such method allows us to construct a graph. Of course, if we want the graph not to have cycles, there should be a large set of prohibitively high costs.

8. Conclusion

When we think about many real life examples (e.g. school/college/internship/etc. matchings), peer effects should be a necessary component of agents preferences. Thus, it seems crucial to be able to identify conditions for existence of a stable matching in the presence of peer effects. Moreover, it is worth being able to explicitly construct a stable matching.

Current paper provides an algorithm, which can be used to construct a (pairwise) stable matching in the presence of peer effects. The sufficient (and in some sense necessary for the existence of a stable matching) condition for the algorithm to work is that the graph, which governs the ability of agents to apply to different schools, does not have cycles (nor directed, nor undirected). The algorithm uses school values and capacities, agents types and their costs associated with applying to different schools, and a peer effect function as inputs. The algorithm takes a finite amount of time, which is polynomial in the number of schools. Therefore, theoretically it is possible for a central planner to implement such mechanism if one has enough information regarding the underlying economy. The resulting stable matching is assortative inside each market (but not across markets). Duflo et al. (2008) show by means of a randomized experiment in Kenya that students benefit from tracking (that is from being assigned to classes assortatively). Thus, we can view the assortative pattern of the outcome of our algorithm as an advantage.

In countries like Russia or China, where government has a power to prohibit groups of people to applying to some schools, the existence result of our paper can be used in market design. Armed with the knowledge that with cycles there may be no stable matching, a market designer can separate schools and students into groups without cycles to ensure the existence of a stable matching.

In case of a decentralized markets, we may view our stable matching as an outcome of a decentralized game between schools and students. As is common for an equilibrium notions, we may get multiple stable matchings. In particular, we do not have a unique stable matching when a peer effect component is very important (i.e. α is large enough). When α is large enough, our model resembles a coordination problem, which is known to have multiple equilibria. In contract, when peer effects are negligible (i.e. $\alpha \approx 0$) we go back to a classical many-to-one matching problem with identical preferences on the schools side, which has a unique solution.

Our algorithm and existence condition rely on the structure of the switching graph. It is still an open question whether we can formulate additional conditions to ensure existense, if we do not have the graph as exogenously given, but start from costs per se. Obviously, we know that if a cost of going from i to j is more than utility from the best outcome in j, then we can erase an edge $\{i \to j\}$. Yet, it may be possible to say something more for intermediate values of costs based on their relative values when compared to feasible utilities even in the presence of cycles. Which intermediate values would guarantee the existence of a stable matching in the presence of cycles? This issue is left for further research.

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9. Appendix

Proof of Theorem 1.

Proof. Let us provide an iterative construction, which leads to a stable matching in a finite time. Then we will show, why it works. We work separately with each tree from the forest. Fix any tree from the forest.

Choose an arbitrary node to be the root of the tree and denote it as m_0. Denote its children as $m_{1,1}, \dots m_{1,z_1}$, where z_1 is the number of children of m_0. Similarly, denote all "grand-children" of m_0 (i.e. children of $m_{1,1}, \dots m_{1,z_1}$) as $m_{2,1}, \dots, m_{2,z_2}$ and so on. That notation is illustrated in the Figure 4.

Now consider the following procedure, where the outside option can be viewed as the worst school with fixed zero utility. Suppose that the longest path from the root to a terminal node (leaf) has K edges.

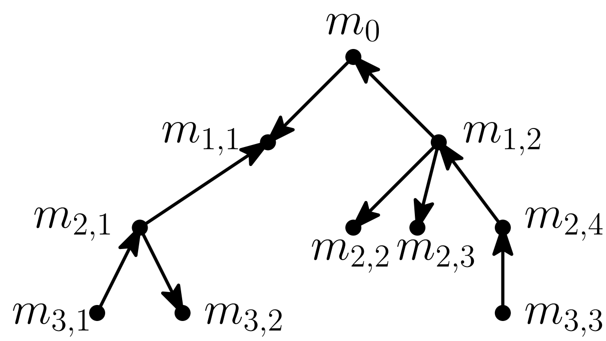

FIGURE 4. Directed tree: notation.

Algorithm:

Step 0**:** Put the best students from all the markets, who can be at m_0, up to capacity to the best school at m_0 (school $v_1^{m_0}$). Note that since, by our assumptions, F_i has a large number of zero types, we are able to fill any school up to capacity. We work with market m_0 and the subset of markets $m_{1,1}, \ldots, m_{1,z_1}$, which are connected to m_0 by an edge directed towards m_0.

Denote by $\bar{\theta}_{k,z}$ the best student from $m_{k,z}$. For each $\bar{\theta}_{1,z}$, who gets a seat at $v_1^{m_0}$, ask what school he/she prefers the most among $v_1^{m_0}$, $v_1^{m_{1,z}}$, and $v_1^{m_{2,z'}}$ for all $m_{2,z'}$ to which one can go from $m_{1,z}$. That is, whether one prefers the above allocation at $v_1^{m_0}$, or allocation, where we put top students from $m_{1,z}$ and its eligible children markets to the best school at $m_{1,z}$, or allocation, where we put top students from $m_{1,z}$ along with all other eligible markets to $m_{1,z}$'s child market $m_{2,z'}$. Similarly ask $\bar{\theta}_0$ (if gets a seat at $v_1^{m_0}$) what school he/she prefers the most among $v_1^{m_0}$ and $v_1^{m_{1,z''}}$ for all markets $m_{1,z''}$ where one can move from m_0.

In the Figure 4 that would mean asking $\bar{\theta}_0$ and $\bar{\theta}_{1,2}$. We ask $\bar{\theta}_0$ whether one prefers seating at m_0 with students from m_0 and $m_{1,2}$ or seating at $m_{1,1}$ with students from m_0, $m_{1,1}$, and $m_{2,1}$. We ask $\bar{\theta}_{1,2}$ whether one prefers seating at m_0 with students from m_0 and $m_{1,2}$, or seating at $m_{1,2}$ with students from $m_{1,2}$ and $m_{2,4}$, or seating at $m_{2,2}$ with students from $m_{1,2}$, and $m_{2,2}$, or seating at $m_{2,3}$ with students from $m_{1,2}$, and $m_{2,3}$.

- If $\bar{\theta}_0$ does not get a seat at some $v_1^{m_{1,z''}}$, move to Step 1;

- If some of such $\bar{\theta}_{1,z}$ does not get a seat at $v_1^{m_{1,z}}$, move to Step 1;
- If some of such $\bar{\theta}_{1,z}$ does not get a seat at $v_1^{m_{2,z'}}$, move to Step 2;
- If $\bar{\theta}_0$ gets a seat everywhere and prefers some $v_1^{m_{1,z''}}$, move to Step 1;
- If some of such $\bar{\theta}_{1,z}$ gets a seat everywhere and prefers $v_1^{m_{1,z}}$, move to Step 1;
- If some of such $\bar{\theta}_{1,z}$ gets a seat everywhere and prefers $v_1^{m_{2,z'}}$, move to Step 2;
- Otherwise fix the above assignment at $v_1^{m_0}$. Delete that school and its students. Go back to Step 0 with the new economy.

Step 1: Consider the market identified at Step 0. Denote it $m_{1,z}$. Put the best students from market $m_{1,z}$ and its eligible to travel to $m_{1,z}$ children and parent markets to the best school at $m_{1,z}$ up to capacity. For each $\bar{\theta}_{2,z'}$, who gets a seat at $v_1^{m_{1,z}}$, ask what school he/she prefers the most among $v_1^{m_{1,z}}$, $v_1^{m_{2,z'}}$, and $v_1^{m_{3,z''}}$ for all $m_{3,z''}$ to which one can go from $m_{2,z'}$. Similarly ask $\bar{\theta}_{1,z}$ (if gets a seat at $v_1^{m_{1,z}}$) what school he/she prefers the most among $v_1^{m_{1,z}}$ and $v_1^{m_{2,z''}}$ for all markets $m_{2,z''}$ where one can move from $m_{1,z}$.

Note that we do not need to ask $\bar{\theta}_0$ even if one gets a seat at $v_1^{m_{1,z}}$. If $\bar{\theta}_0$ gets a seat at $v_1^{m_{1,z}}$, and we get that market from previous step, then it was $\bar{\theta}_0$'s first choice. Similarly, we do not need to ask $\bar{\theta}_{1,z}$ about m_0. If it is possible to travel from $m_{1,z}$ to m_0 and we get $m_{1,z}$ from Step 1, it means either $\bar{\theta}_{1,z}$ does not get a seat at $v_1^{m_{1,z}}$, so we do not ask $\bar{\theta}_{1,z}$ at all, or it is $\bar{\theta}_{1,z}$'s first choice, thus, it is better than m_0.

- If $\bar{\theta}_{1,z}$ does not get a seat at some $v_1^{m_{2,z''}}$, move to Step 2;
- If some of such $\bar{\theta}_{2,z'}$ does not get a seat at $v_1^{m_{2,z'}}$, move to Step 2;
- If some of such $\bar{\theta}_{2,z'}$ does not get a seat at $v_1^{m_{3,z^*}}$, move to Step 3;
- If $\bar{\theta}_{1,z}$ gets a seat everywhere and prefers some $v_1^{m_{2,z''}}$, move to Step 2;
- If some of such $\bar{\theta}_{2,z'}$ gets a seat everywhere and prefers $v_1^{m_{2,z'}}$, move to Step 2;
- If some of such $\bar{\theta}_{2,z'}$ gets a seat everywhere and prefers $v_1^{m_{3,z^*}}$, move to Step 3;
- Otherwise fix the above assignment at $v_1^{m_{1,z}}$. Delete that school and its students. Go back to Step 0 with the new economy.

...

Step k: Do the same thing as in the previous steps, but with the best school at market $m_{k,z}$, $v_1^{m_{k,z}}$. It is the market, which we identified in previous steps (either at Step $k-1$ or at Step $k-2$). Put the best students from market $m_{k,z}$ and its eligible to travel to $m_{k,z}$ children and parent markets to the best school at $m_{k,z}$ up to capacity. For each $\bar{\theta}_{k+1,z'}$, who gets a seat at $v_1^{m_{k,z}}$, ask what school he/she prefers the most among $v_1^{m_{k,z}}$, $v_1^{m_{k+1,z'}}$, and $v_1^{m_{k+2,z''}}$ for all $m_{k+2,z''}$ to which one can go from $m_{k+1,z'}$. Similarly ask $\bar{\theta}_{k,z}$ (if gets a seat at $v_1^{m_{k,z}}$) what school he/she prefers the most among $v_1^{m_{k,z}}$ and $v_1^{m_{k+1,z''}}$ for all markets $m_{k+1,z''}$ where one can move from $m_{k,z}$.

As before, we do not need to ask the parent of $\bar{\theta}_{k,z}$ even if one gets a seat at $v_1^{m_{k,z}}$. If the parent gets a seat at $v_1^{m_{k,z}}$, and we get market $m_{k,z}$ from Step $k-1$ or $k-2$, then $v_1^{m_{k,z}}$ was the parent's first choice. Similarly, we do not need to ask $\bar{\theta}_{k,z}$ about its parental market. If it is possible to travel from $m_{1,z}$ to the parental market and we get $m_{k,z}$ from previous steps, then it must be from Step $k-1$ (with such edge direction $m_{k,z}$ does not participate in Step $k-2$). Thus, either $\bar{\theta}_{k,z}$ does not get a seat at $v_1^{m_{k,z}}$, so we do not ask $\bar{\theta}_{k,z}$ at all, or it is $\bar{\theta}_{k,z}$'s first choice, thus, it is better than the parental market.

- If $\bar{\theta}_{k,z}$ does not get a seat at some $v_1^{m_{k+1,z''}}$, move to Step $k+1$;
- If some of such $\bar{\theta}_{k+1,z'}$ does not get a seat at $v_1^{m_{k+1,z'}}$, move to Step $k+1$;
- If some of such $\bar{\theta}_{k+1,z'}$ does not get a seat at $v_1^{m_{k+2,z^*}}$, move to Step $k+2$;
- If $\bar{\theta}_{k,z}$ gets a seat everywhere and prefers some $v_1^{m_{k+1,z''}}$, move to Step $k+1$;
- If some of such $\bar{\theta}_{k+1,z'}$ gets a seat everywhere and prefers $v_1^{m_{k+1,z'}}$, move to Step $k+1$;
- If some of such $\bar{\theta}_{k+1,z'}$ gets a seat everywhere and prefers $v_1^{m_{k+2,z^*}}$, move to Step $k+2$;
- Otherwise fix the above assignment at $v_1^{m_{k,z}}$. Delete that school and its students. Go back to Step 0 with the new economy.

...

Step K: We must stop if we have reached a node $m_{K,z}$, as by definition it is a terminal node. No other markets can get a seat at $v_1^{m_{K,z}}$, thus all bullets except the last one in the above steps are not satisfied, and we are left with the last bullet point, i.e. we finalize the assignment.

Let us explain why the algorithm, presented above, leads to a stable matching. Note that in each step we are trying to get the best possible scenario for the highest type in some market. Thus, that type does not want to deviate: schools in a given market by construction have decreasing peer effect and value, thus, there is no reason to deviate to a school with a larger number in the same market. Here we are using the properties of a peer effect function, which implies that if peers in one set are weakly larger than in the other, then the former set has weakly higher value of a peer effect function. Moreover, there is no reason to deviate to the other possible market, as in the algorithm we were choosing the best market.

We also need to show that agents, which are assigned to some school during some step in the algorithm, and were not the highest types in that step, still do not want to deviate. Suppose we implement an assignment at Step k, that is, we fill a school at some market $m_{k,z}$. Thus, if $\bar{\theta}_{k,z}$, its parental market $\bar{\theta}_{k-1,z'}$, and any of its children markets $\bar{\theta}_{k+1,z''}$ get a seat at $v_1^{m_{k,z}}$, then it is their desired allocation. (They get a seat in all of the markets, where they are eligible to travel, but choose $m_{k,z}$.) Let us look at the second highest type from $m_{k,z}$, θ'. Staying in the same school as $\bar{\theta}_{k,z}$, θ' gets higher utility, as its set of peers is better:

$$peers(\theta') = peers(\bar{\theta}_{k,z}) \cup \{\bar{\theta}_{k,z}\} \setminus \{\theta'\}.$$

Moreover, deviating to a different market leads to a weakly lower utility than $\bar{\theta}_{k,z}$ was getting, while we were doing a comparison at Step k (or $k-1$ or $k-2$). Deviating to the best school at the other market means sharing weakly worse set of peers than $\bar{\theta}_{k,z}$ had: $\bar{\theta}_{k,z}$ is no longer there and is replaced by someone worse. If θ' was at that school with $\bar{\theta}_{k,z}$, then even by someone weakly worse than θ' and the best possible set of peers is $peers(\bar{\theta}_{k,z})$ from that school at the moment of comparison at Step k (or $k-1$ or $k-2$). If θ' was not at that school with $\bar{\theta}_{k,z}$, then he takes $\bar{\theta}_{k,z}$'s place and, again, gets peers no better than $\bar{\theta}_{k,z}$ had. Thus, deviating to a different market leads to a weakly smaller utility than $\bar{\theta}_{k,z}$ had at that market, while staying with $\bar{\theta}_{k,z}$ leads to a weakly higher utility than $\bar{\theta}_{k,z}$ has.

Thus, second highest type from $m_{k,z}$ does not deviate. Similarly, other agents from $m_{k,z}$, $m_{k-1,z'}$, and $m_{k+1,z''}$ do not deviate. $\square$

Lemma 1. *The algorithm in Theorem 1 does not depend on the choice of the starting market.*

Proof. Suppose we start from some market m^* and first fill a school v_ℓ^i with highest types from markets $\{m_1, \ldots, m_r\}$. Let us explain that if we start with a different market, v_ℓ^i will still be filled with the same agents. Because we assigned highest types from markets $\{m_1, \ldots, m_r\}$ to v_ℓ^i at the first step of the algorithm, when starting from m^*, it must be that school v_ℓ^i filled with the best possible peers is the best option for $\bar{\theta}_{m_1}, \ldots, \bar{\theta}_{m_r}$, where $\bar{\theta}_{m_j}$ denotes the maximal type in F_{m_j}. Thus, when we start from a different market, we can not assign any of $\bar{\theta}_{m_1}, \ldots, \bar{\theta}_{m_r}$ to another school, as they will be asked to compare that other option with v_ℓ^i filled with the best possible peers, and they will choose the latter. Similarly, we can not fill v_ℓ^i with other agents, as until we assign $\bar{\theta}_{m_1}, \ldots, \bar{\theta}_{m_r}$ somewhere, they represent a subset of peers admitted to v_ℓ^i. The same reasoning applies to schools filled later in the algorithm, which starts at m^*. $\square$

Proof of Theorem 2.

Proof. Suppose G has a cycle. Choose the smallest cycle of G. Without loss of generality let us assume that it involves markets $1, \ldots, \ell$.

We now specify parameters (number of schools per market, their qualities and capacities, switching costs between markets, and types of students). Let us fix $k_i = 1$, $q_1^i = 2$ for all markets i. That is, there is only one school per market with capacity 2.

The further procedure splits into two parts: we separately make a specification for the cycle markets $1, \ldots, \ell$ and for the outside markets $\ell + 1, \ldots, n$. First, let us show that for any choices for the cycle markets, we can make complimentary choices for the outside markets so that the matching problem splits into two independent ones: for the cycle markets and for the rest.

Indeed, choose all switching costs c_{ij} with either $i > \ell$ or $j > \ell$ in an arbitrary way. Further, set for $i > \ell$, $F_i = \{M_i, M_i\}$, where M_i is an increasing sequence of i, such that $M_{\ell+1}$ is strictly larger than types of all agents in markets $1, \ldots, \ell$. That is, the higher is the number of a market, the better students occupy it, and all the students in the outside markets are better than those in the cycle markets. Next, for $i > \ell$, we set v_1^i to be an increasing function of i such that for each $i > j > \ell$, $v_1^i > v_1^j + \alpha p(M_n)$ and $v_1^{\ell+1}$ is larger than qualities of all schools in markets $1, \dots, \ell$. This condition guarantees that it is better to be alone at school v^i, $i > \ell$ than to go to a worse school v^j, $j < i$, with the best possible peer. Under the above conditions, in any stable matching we must have that agents from market n, $\{M_n, M_n\}$ stay home and attend v_1^n. Further, agents from market $n-1$ also stay home and attend v_1^{n-1} and so on until market $\ell + 1$. We conclude that the matching for the markets $\ell + 1, \dots, n$ is independent from the one for the markets $1, \dots, \ell$. Therefore, for the rest of the proof we can and will consider only the markets $1, \dots, \ell$.

The remaining proof consists of three separate cases depending on the structure of the cycle.

Case I (directed cycle): First suppose that the cycle is directed, so that (without loss of generality) it has the following structure: $1 \to 2 \to \cdots \to \ell \to 1$. Since our cycle is the smallest, there are no other links between the first ℓ vertices of G.

Take $M > \ell + 1$ to be specified later, and consider the following sets of types: $F_1 = \{M, 0\}, F_2 = \{M-1, M-3\}, \dots, F_\ell = \{M - \ell + 1, M - \ell - 1\}$[6]. Therefore, in any market $1 < i < \ell$ there is one type greater than any one in F_{i+1} and one type which lies between types in F_{i+1}.

We specify switching costs $c_{i,i+1}$, $i = 1, 2, \dots, \ell$,[7] so that

$$\alpha(p(M) - p(M - \ell - 1)) < c_{i,i+1} < \alpha(p(M - \ell - 1) - p(0)). \tag{1}$$

Note that the growth condition on $p(\cdot)$ given in Definition 10 guarantees that if M is large enough, then such a choice of costs $c_{i,i+1}$ always exists. Next choose arbitrary $v > \max_{i=1,\dots,\ell} c_{i,i+1}$ and set $v_1 = \cdots = v_\ell = v$.

[6]We do not explicitly write additional zero types in each F_i, as they will not play a role in the argument. In fact, throughout the proof we ignore their existence.

[7]For ease of notation we avoid writing $i \, mod \, \ell$ and assume that $i = \ell + 1$ stands for market 1 in the remaining proof.

This finishes the specification. We now prove that no stable matching exists. To begin with, notice that the first inequality in (1) guarantees that for each agent it is better to be matched in the home market with any peer except 0, rather than to go to another market: in the home market i the utility for the agent from this market is at least $\alpha p(M-\ell-1)+v$ and in the market $i+1$ the utility of the agent from i is at most $\alpha p(M)+v-c_{i,i+1}$, which is smaller.

Second, we show that it is impossible for both agents from market i to be assigned to market $i+1$. Indeed, if this happens, then the higher type born in market $i+1$ would prefer to stay in his home market, and therefore he will push away the lower type born in market i from the market $i+1$.

Third, we are going to show that in any stable matching all agents, except, perhaps, the agent 0 from the market 1, must be matched to some school (rather than choose an outside option with zero utility). Choose a market i. If no agent from market $i-1$ is assigned to i, then there is nothing to prove (both agents from $i+1$ can stay at home, which is better than taking outside option). The only remaining case is that there is exactly one agent from market $i-1$, who is assigned to i. In this case, each of the agents from market i has two options which are both better than the outside option (by our choice of v): either to take the second seat in the home-school, or to move to the school in the market $i+1$. If $i\neq 1$, then our choice of types implies that any one agent from i will always be accepted to the school in market $i+1$, if he prefers to go there. Hence, in this case both agents from market i should not choose the outside option. The case $i=1$ needs a separate consideration because of type 0 there — but we do not claim anything for the agent 0.

Forth, let us show that the autarky allocation with each agent staying in his own market is not stable. Indeed, suppose we have autarky allocation. In that case M will deviate to market 2 to get a better peer: $c_{12}<\alpha(p(M-3)-p(0))$. Similarly, even if 0 remains unassigned, M will deviate to the second market.

At this point there is only one remaining possibility: for some $m=1,2,\dots,\ell$, one agent from the market m moves to market $m+1$, one agent from market $m+1$

moves to market $m+2,\ldots$, one agent from market ℓ moves to market 1. All other agents stay in their own markets.

Suppose $m > 1$. Then the agent who moved from market m to market $m+1$ should deviate: there is one empty seat in market m. The second seat at m is occupied by the other agent from m, who has non-zero type. Thus, the former agent can take the remaining empty seat in the school of his home market, and we have shown above that matching to a schools in one's own market with a non-zero peer is preferable for each agent.

Thus, we must have $m = 1$. In that case either type 0 goes to v_1^2 along with one of the market 2's agents or 0 stays at home and M moves to market 2. If 0 moves, then the agent from market 2 who is assigned to v_1^3 will deviate and not pay costs, since $c_{23} > \alpha(p(M-2) - p(M-3))$. If 0 stays at home, then he is joined by one of the agents from market ℓ. Yet then M will deviate home, since $c_{12} > \alpha(p(M-1) - p(M-\ell-1))$. Thus, there is no stable matching for the constructed configuration.

Case II (undirected cycle): Now suppose that the above cycle is not a directed cycle. There are two possible subcases. Either there is only one vertex with both edges going away from it or at least two of them. Such type of vertex is shown in the Figure 5. Note that the case with non such vertices corresponds to a directed cycle.

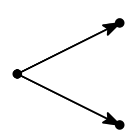

FIGURE 5. Vertex with both edges going away from it.

(1) Suppose that there is only one vertex with both edges going away from it. Then the cycle is shown in the Figure 6[8]. That is, there are two directed

[8]For convenience we have changed the names of markets $i_1, \ldots, i_\ell$.

pathes from A to B: one via A_i's and the other via B_j's. Note that a non-directed cycle must have at least three vertices. Those necessary three vertices are $\{A, B, B_1\}$.

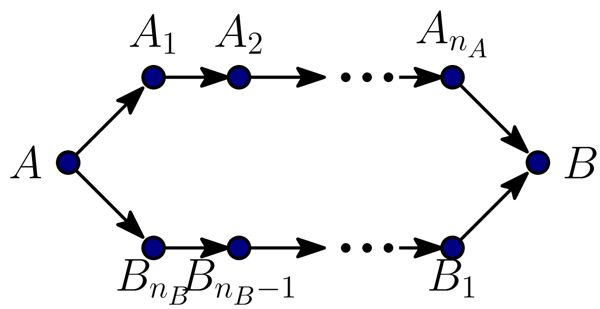

FIGURE 6. Case II.1.

Now let us define the market structure. Choose M large enough and consider the following sets of types[9]:

$$F_A := \{\theta_A, \theta_A\} = \{M - n_A - 1, M - n_A - 1\},$$

$$F_{A_i} := \{\theta_{A_i}, \theta_{A_i}\} = \{M - (n_A + 1 - i), M - (n_A + 1 - i)\},$$

$$F_B = \{0, 0\},\ F_{B_1} := \{\theta_{B_1}, 0\} = \{M, 0\},$$

$$F_{B_i} := \{\theta_{B_i}, \theta_{B_i}\} = \{M - n_A - 2 - (n_B - i), M - n_A - 2 - (n_B - i)\},\ i \neq 1.$$

That is, the highest type lives in B_1. The next highest types are in A_{n_A}, and with the decrease in A's subscript, types decline, until we reach the last one, A_1. Among the remaining markets, the highest types are in A, followed by B_{n_B}. With the decrease in B's subscript, types decline until we reach B_2.

Now let us define values of schools:

$$v_1^A = 0,\ v_1^{B_1} = \ldots = v_1^{B_{n_B}} \equiv v,$$

$$v_1^{A_1} = v + \alpha p(M) + 1,\ v_1^{A_2} = v_1^{A_1} + \alpha p(M) + 1 = v + 2\alpha p(M) + 2, \ldots,$$

$$v_1^{A_i} = v_1^{A_{i-1}} + \alpha p(M) + 1 = v + i\alpha p(M) + i, \ldots,$$

$$v_1^B = v_1^{A_{n_A}} + \alpha p(M) + 1 = v + (n_A + 1)\alpha p(M) + n_A + 1.$$

[9]Again, throughout the proof we ignore the existence of additional zeros.

Thus, in the markets $A_1, \dots, A_{n_A}, B$ values are highest and they are an increasing sequence.

Assume all switching costs to be smaller than v, so that it is always better to be assigned to a foreign school than stay unassigned. Moreover, choose costs such that

$$v_1^{A_{n_A}} + \alpha p(M) < v_1^B - c_{A_{n_A}B} + \alpha p(0) \Leftrightarrow c_{A_{n_A}B} < 1 + \alpha p(0). \tag{2}$$

$$v_1^{A_i} + \alpha p(M) < v_1^{A_{i+1}} - c_{A_iA_{i+1}} + \alpha p(0) \Leftrightarrow c_{A_iA_{i+1}} < 1 + \alpha p(0),\ i < n_A. \tag{3}$$

$$\begin{aligned} &\begin{cases} v_1^{B_1} + \alpha p(0) < v_1^B + \alpha p(\theta_{A_{n_A}}) - c_{B_1B}, \\ v_1^{B_1} + \alpha p(\theta_{B_2}) < v_1^B + \alpha p(\theta_{A_{n_A}}) - c_{B_1B} \end{cases} \\ \Leftrightarrow &\begin{cases} c_{B_1B} < (n_A+1)\alpha p(M) + n_A + 1 + \alpha(p(\theta_{A_{n_A}}) - p(0)), \\ c_{B_1B} > (n_A+1)\alpha p(M) + n_A + 1 + \alpha(p(\theta_{A_{n_A}}) - p(\theta_{B_2})). \end{cases} \end{aligned} \tag{4}$$

Moreover, choose $c_{AA_1} = c_{AB_{n_B}} < v$, so that agent θ_A prefers A_1 over B_{n_B} (and both over A). This is because $v_1^{A_1} + \alpha p(0) = v + \alpha p(M) + 1 + \alpha p(0) > v + \alpha p(M) = v_1^{B_{n_B}} + \alpha p(M)$. Additionally choose $c_{B_iB_{i-1}}$, $i > 1$ such that

$$\begin{aligned} &\alpha(p(\theta_{B_i}) - p(0)) < c_{B_iB_{i-1}} < v \\ &\Leftrightarrow \alpha(p(M - n_A - 2 - (n_B - i)) - p(0)) < c_{B_iB_{i-1}} < v^{10}. \end{aligned} \tag{5}$$

Eq. (2) guarantees that at least one agent from A_{n_A} switches to B. Whether the second agent switches depends on the behavior of agent with highest possible type, M, from market B_1. Eq. (4) guarantees that θ_{B_1} prefers to switch to B and enjoy the peer effect from $\theta_{A_{n_A}}$ instead of staying with zero type at home. However, θ_{B_1} will not switch to B if θ_{B_2} joins him at B_1. Eq.

[10]Existence of such $c_{B_iB_{i-1}}$ can be guaranteed by choosing v high enough.

(5) guarantees that all agents from markets $B_i, i > 1$ will stay at home even with zero type of peer.

Suppose first that we have a stable matching where agent $\theta_{B_1} = M$ from B_1 goes to B. Then only one agent can switch from A_{n_A}. The second agent stays at A_{n_A}. By Eq. (3), we know that one agent switches from A_{n_A-1} to A_{n_A}, then one agent switches from A_{n_A-2} to A_{n_A-1} and so on, until A_1. Moreover, as agents in A prefer A_1 over B_{n_B}, one agent from A will switch to A_1. The second one is forced to go to B_{n_B} (it is better than staying at A). Thus one of agents from B_{n_B} is pushed away. That agent moves to B_{n_B-1}, as it is better to be assigned than unassigned. Similarly, we end up with agent θ_{B_2}, who is pushed to B_1. Thus, by Eq. (4), type θ_{B_1} deviates and moves back to B_1.

Now suppose that we have a stable matching where agent $\theta_{B_1} = M$ from B_1 does not go to B and stays at home. Thus, by Eq. (2) both agents from A_{n_A} switch to B. Similarly both agents from A_{n_A-1} switch to A_{n_A} and so on. Finally, both agents from A switch to A_1. By Eq. (4), type θ_{B_1} will deviate to B, if his peer is zero type. Therefore, he must be with θ_{B_2}. However, as no one from market A goes to B_{n_B} both agents from B_{n_B} are not pushed away from home. Thus, by Eq. (5), they stay at B_{n_B}. Similarly, by Eq. (5) all agents θ_{B_i}, $i > 1$ stay at home. Thus, θ_{B_2} cannot be at B_1 with θ_{B_1}, and we get a contradiction. Therefore, no stable matching exists for the above economy.

(2) Suppose that there are at least two vertices with both edges going away from them. Then the cycle is shown in the Figure 7[11]. That is, nodes with names A_{2k-1} have both edges pointing away from them, nodes with names A_{2k} have both edges pointing towards them, nodes with names B_i^{2k-1} have edges both in the direction of A_{2k} (first towards then away), and nodes with names B_i^{2k} have edges both in the direction of A_{2k} (first away then towards).

[11]For convenience we have changed the names of markets $i_1, \dots, i_\ell$.

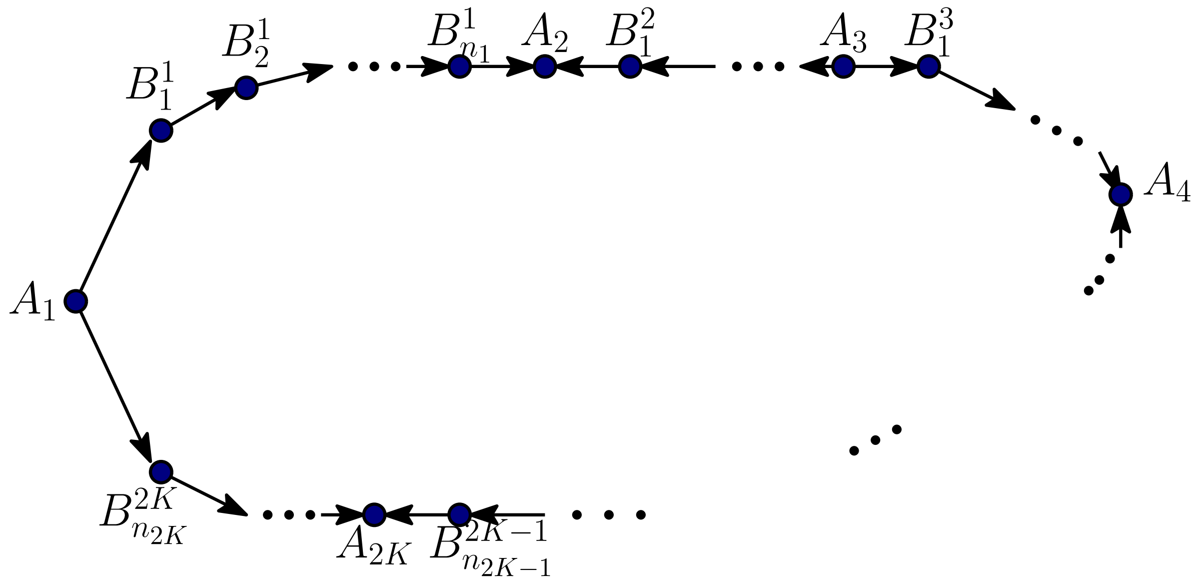

FIGURE 7. Case II.2.

Now let us define the market structure. Choose M large enough, so that $2p(M-K) > p(M)$[12] and consider the sets of types[13]:

$$F_{A_1} := \{\theta_{A_1}, 0\} = \{M, 0\},\; F_{A_{2l}} = \{0, 0\}\,\forall l,$$

$$F_{A_{2l+1}} := \{\theta_{A_{2l+1}}, \theta_{A_{2l+1}}\} = \{M - l, M - l\},\; l > 0$$

$$F_{B_i^{2l+1}} := \{\theta_{B_i^{2l+1}}, \theta_{B_i^{2l+1}}\} = \left\{\theta_{A_{2l+1}} - \frac{i}{1 + n_{2l+1}}, \theta_{A_{2l+1}} - \frac{i}{1 + n_{2l+1}}\right\}$$

$$= \left\{M - l - \frac{i}{1 + n_{2l+1}}, M - l - \frac{i}{1 + n_{2l+1}}\right\},$$

$$F_{B_i^{2l}} := \{\theta_{B_i^{2l}}, \theta_{B_i^{2l}}\} = \left\{\theta_{A_{2l+1}} + \frac{1 + n_{2l} - i}{(1 + n_{2l})(1 + n_{2l-1})}, \theta_{A_{2l+1}} - \frac{1 + n_{2l} - i}{(1 + n_{2l})(1 + n_{2l-1})}\right\}$$

$$= \left\{M - l + \frac{1 + n_{2l} - i}{(1 + n_{2l})(1 + n_{2l-1})}, M - l + \frac{1 + n_{2l} - i}{(1 + n_{2l})(1 + n_{2l-1})}\right\},\; i \neq K,$$

$$F_{B_i^{2K}} := \{\theta_{B_i^{2K}}, \theta_{B_i^{2K}}\} = \left\{\theta_{A_{2K-1}} - 1 - \frac{i}{1 + n_{2K}}, \theta_{A_{2K-1}} - 1 - \frac{i}{1 + n_{2K}}\right\}$$

$$= \left\{M - K - \frac{i}{1 + n_{2K}}, M - K - \frac{i}{1 + n_{2K}}\right\}.$$

[12]This can be done as $p(\cdot)$ grows slower than exponentially.

[13]Again, throughout the proof we ignore the existence of additional zeros.

That is, the highest type lives in A_1. The larger is the subscript of A_i (among odd ones), the smaller are types in that market. Moreover, $\theta_{B^{2l+1}_{n_{2l+1}}} > \theta_{B^{2l+2}_1}$, and types in intermediate markets, B^k_i are decreasing in i and they lie on the intervals $(\theta_{A_k} - 1, \theta_{A_k})$ for odd k and $(\theta_{A_{k+1}}, \theta_{A_{k+1}} + 1)$ for even $k \neq 2K$.

Now let us define values of schools:

$$v_1^{A_{2l+1}} = 0,\ v_1^{B_i^{2l+1}} = v_1^{A_{2l}} \equiv v\,\forall l, i,$$

$$v_1^{B_i^{2l}} = (n_{2l} - i + 1)\alpha p(M - l + 1) + n_{2l} - i + 1\,\forall l, i,$$

where v is large enough so that $v > (n_{2l} + 1)\alpha p(M - l + 1) + n_{2l} + 1\,\forall l$.

Assume all switching costs to be smaller than v, so that it is always better to be assigned to a foreign school than stay unassigned. Set $c_{A_1 B^{2K}_{n_{2K}}} = 0$. Moreover, choose costs such that (assume that $B^{2l+1}_{n_{2l+1}+1} = A_{2l+2}$ and $B^{2l}_0 = A_{2l}$)

(6)
$$v_1^{B^{2l}_{i-1}} + \alpha p(0) - c_{B^{2l}_i B^{2l}_{i-1}} > v_1^{B^{2l}_i} + \alpha p\left(M - l - \frac{1 + n_{2l} - i}{(1 + n_{2l})(1 + n_{2l-1})}\right)$$
$$\Rightarrow c_{B^{2l}_i B^{2l}_{i-1}} < \alpha\left(p(M - l + 1) - p\left(M - l - \frac{1 + n_{2l} - i}{(1 + n_{2l})(1 + n_{2l-1})}\right) + p(0)\right) + 1,$$

(7)
$$v_1^{B^{2l}_{n_{2l}}} + \alpha p(0) - c_{A_{2l+1} B^{2l}_{n_{2l}}} > v_1^{A_{2l+1}} + \alpha p(M - l) = \alpha p(M - l)$$
$$\Rightarrow c_{A_{2l+1} B^{2l}_{n_{2l}}} < \alpha\left(p(M - l + 1) - p(M - l) + p(0)\right) + 1,$$

(8)
$$v_1^{B_1} + \alpha p\left(M - \frac{1}{1 + n_1}\right) - c_{A_1 B^1_1} > v_1^{B^{2K}_{n_{2K}}} + \alpha p(0) - c_{A_1 B^{2K}_{n_{2K}}}$$
$$\Rightarrow \begin{cases} c_{A_1 B^1_1} < v - \alpha p(M - K - 1) - 1 + \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p(0), & n_{2K} \neq 0, \\ c_{A_1 B^1_1} < \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p(0), & n_{2K} = 0 \end{cases},$$

(9)
$$v_1^{B_1} + \alpha p\left(M - \frac{1}{1 + n_1}\right) - c_{A_1 B^1_1} < v_1^{B^{2K}_{n_{2K}}} + \alpha p\left(M - K - \frac{n_{2K}}{1 + n_{2K}}\right) - c_{A_1 B^{2K}_{n_{2K}}}$$
$$\Rightarrow \begin{cases} c_{A_1 B^1_1} > v - \alpha p(M - K - 1) - 1 + \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p\left(M - K - \frac{n_{2K}}{1+n_{2K}}\right), & n_{2K} \neq 0, \\ c_{A_1 B^1_1} > \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p(M - K + 1), & n_{2K} = 0, \end{cases},$$

$$v_1^{B_i^{2l+1}} + \alpha p(0) > v_1^{B_{i+1}^{2l+1}} + \alpha p\left(M - l - \frac{i}{1+n_{2l+1}}\right) - c_{B_i^{2l+1} B_{i+1}^{2l+1}}$$
$$\Leftrightarrow c_{B_i^{2l+1} B_{i+1}^{2l+1}} > \alpha\left(p\left(M - l - \frac{i}{1+n_{2l+1}}\right) - p(0)\right). \tag{10}$$

Note that Eq. (9) and Eq. (10) do not contradict the initial requirement that $c < v$, as because $p(\cdot)$ grows slower than exponentially, we can find M such that $2p(M-K) > p(M)$ and $v > v - \alpha p(M-K-1) - 1 + \alpha p\left(M - \frac{1}{1+n_1}\right) - \alpha p\left(M - K - \frac{n_{2K}}{1+n_{2K}}\right)$. Moreover as $v_1^{B_i^{2l+1}} = v$, by choice of v, $v_1^{B_i^{2l+1}} > (n_{2l}+1)\alpha p(M-l+1) + n_{2l} + 1 > \alpha\left(p\left(M - l - \frac{i}{1+n_{2l+1}}\right) - p(0)\right)$.

Because $v_1^{A_1} = 0$, θ_{A_1} will not stay at home in a stable matching. Suppose first that in a stable matching θ_{A_1} goes to B_1^1. Then he pushes away $\theta_{B_1^1}$. Note that the other $\theta_{B_1^1}$ will stay at home, as home guarantees a better peer and no switching costs, while school value is the same at home and abroad. Thus, only one $\theta_{B_1^1}$ goes to B_2^1. Similarly, only one $\theta_{B_2^1}$ goes to B_3^1 and so on. Finally, only one $\theta_{B_{n_1}^1}$ goes to A_2. By Eq. (6) both $\theta_{B_1^2}$ want to switch to A_2, but only one can, as $\theta_{B_{n_1}^1} > \theta_{B_1^2}$. Thus, one $\theta_{B_1^2}$ switches and another stays at home. Similarly, one $\theta_{B_2^2}$ switches to B_1^2 and another stays at home and so on. Thus, one θ_{A_3} switches to $B_{n_2}^2$ and another to B_1^3 and so on. Finally, we are left with the fact that one $\theta_{B_{n_{2K}}^{2K}}$ switches to $B_{n_{2K}-1}^{2K}$ and another stays at home. Then by Eq. (9), θ_{A_1} deviates and switches to $B_{n_{2K}}^{2K}$.

Now suppose that θ_{A_1} goes to $B_{n_{2K}}^{2K}$. Then by Eq. (10), no one switches from B_1^1. Similarly no one switches from B_2^1 and so on until $B_{n_1}^1$. Thus, by Eq. (6), both agents switch from B_1^2 to A_2. By the same logic both agents switch from B_2^2 to B_1^2 and so on. Finally, by Eq. (7), both agents switch from A_3 to $B_{n_2}^2$. Applying the same arguments to next markets, we get that agents from $B_{n_{2K-1}}^{2K-1}$ stay at home, while agents from B_1^{2K} switch to A_{2K} and, at the end, agents from $B_{n_{2K}}^{2K}$ switch to $B_{n_{2K}-1}^{2K}$. Thus, θ_{A_1} is left alone at $B_{n_{2K}}^{2K}$. However, by Eq. (8), in that case θ_{A_1} deviates and switches to B_1^1. Thus, there is no stable matching for the constructed configuration.

□

Proof of Proposition 4.

Proof. In the proof we assume that no two agents share the same type θ, and no agent is indifferent between two schools. That is, first, there do not exist agents θ from market i and θ' from market j such that $\theta = \theta'$, and, second, there do not exist a triple of markets i, j, f and two schools (ℓ, i) and (k, j) such that $v_\ell^i - c_{fi} = v_k^j - c_{fj}$.

Denote the equilibrium from the student-proposing (thus, student-optimal) Gale-Shapley algorithm as eq_1, and suppose there exists another equilibrium eq_2. Let us prove that $eq_2 = eq_1$. When students propose to their most preferred schools, schools start to accept the highest types. The agent with $\theta = \max\{F_1 \cup \ldots \cup F_n\}$ is admitted for sure (this is the most preferred type for schools). Then the second highest type is admitted for sure and so on until some school reaches its capacity. Suppose school v_1^i reaches its capacity first (then those, who were proposing to that school and were not accepted need to propose to another school). Denote by $\underline{\theta}_1^i$ the smallest type admitted to v_1^i.

We claim that all types $\theta \geq \underline{\theta}_1^i$ have the same allocations under both equilibria. Suppose by contradiction, that there exist a type θ_0 from market i_0, such that he has different allocations in eq_1 and eq_2. Under eq_1 he is admitted to the first best option, $v_1^{i(\theta_0)}$, thus, under eq_2 he is admitted to a different school. If he can switch to $v_1^{i(\theta_0)}$, he would do so. Thus, he must be below the cut-off for $v_1^{i(\theta_0)}$, and the school must be filled up to capacity. Thus, $\theta_0 < \underline{\eta}_1^{i(\theta_0)}$, where $\underline{\eta}$ denote cut-offs under eq_2. If he is not accepted, it means that there are other agent, which have types above $\underline{\eta}_1^{i(\theta_0)}$ and are assigned to $v_1^{i(\theta_0)}$, while were assigned to a different school under eq_1. Thus, we get an agent θ_1 from market i_1 who prefers to switch to his first best from eq_1, $v_1^{i(\theta_1)}$, but is assigned to $v_1^{i(\theta_0)}$. Thus, his first best school is filled up to capacity and has threshold $\underline{\eta}_1^{i(\theta_1)} > \theta_1$. Because number of schools is finite, we can repeat the process until we get $k, \ell, k > \ell$ such that $\theta^k > \theta^\ell$, and the first best option for the agent θ^k from market $i(k)$ is $v_1^{i(\theta_k)}$, but he is assigned to a different school $\left(v_1^{i(\theta_{n-1})}\right)$, while θ^ℓ wants to go to $v_1^{i(\theta_\ell)}$ but is admitted to $v_1^{i(\theta_{\ell-1})} = v_1^{i(\theta_k)}$ under eq_2. Thus, θ^k can profitably deviate, and we get a contradiction. Therefore, all types $\theta \geq \underline{\theta}_1^i$ have the same allocations under both equilibria. Applying the same procedure to the next cut-off in eq_1, we get that, again, allocations coincide.

Because number of schools, and, thus, cut-offs is finite, we get that $eq_1 = eq_2$. Thus, the equilibrium is unique. □

Proof of Proposition 5.

Proof. Let us proceed in the following way: first, reorder in any way schools in each market, then apply the algorithm from Theorem 1 for the new school ordering. We will get a different matching, because now not schools with highest v's are getting the best peers in each market, but some other schools. For example, if we reorder schools from last to first, across each market we will see the best students in the schools with the smallest values.

Let us show that no one will deviate from the above assignment. Because v's and c'c now do not matter, no one is going to deviate across markets (all schools, which can accept a given agent have lower peers than one's current assignment). Thus, the only possible deviation would be to a different market. However, the construction in Theorem 1 guarantees that it is also non-profitable. □

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